Control Systems I

Loop Shaping

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Loop Shaping

Goal

Design a controller to achieve a set of specifications on the closed-loop system

Challenge

Closed-loop transfer functions are a highly nonlinear function of the control law

$$\mathcal{T} = \frac{GK}{1 + GK} \qquad \mathcal{S} = \frac{1}{1 + GK}$$

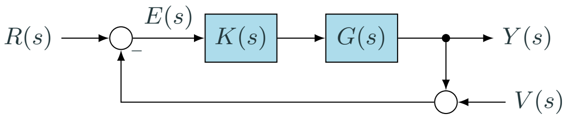
Idea

Define closed-loop characteristics in terms of open-loop response GK.

Shaping the response GK is ***linear*** in K, and much easier

Sensitivity & Complementary Sensitivity

Recall: Closed-Loop Transfer Functions



Two quantities that define the performance of the system:

- Response of error $E(s)$ to output noise $V(s)$

$$\mathcal{S}(s) := \frac{E(s)}{V(s)} = \frac{1}{1 + G(s)K(s)}$$ Sensitivity function

- Response of output $Y(s)$ to reference $R(s)$

$$\mathcal{T}(s) := \frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$ Complementary sensitivity function

Sensitivity & Complementary Sensitivity Functions

Sensitivity Function

$$\mathcal{S}(s) = \frac{E(s)}{V(s)} = \frac{1}{1 + G(s)K(s)}$$

- Impact of noise on the error
- Ideal value : 0

Complementary Sensitivity Function

$$\mathcal{T}(s) = \frac{Y(s)}{Y_c(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

- Impact of reference on the output
- Ideal value : 1

Functions are *complementary*:

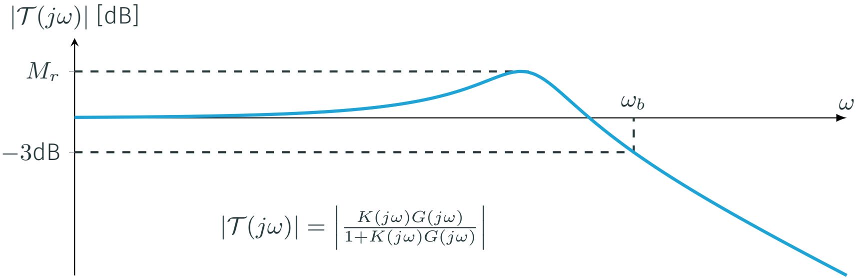
$$\mathcal{S}(s) + \mathcal{T}(s) = \frac{1}{1 + G(s)K(s)} + \frac{G(s)K(s)}{1 + G(s)K(s)} = 1$$

Changes in one will cause changes in the other - limits of performance

The sensitivity and complementary sensitivity functions are transfer functions:

- We can compute their frequency responses: $\mathcal{S}(j\omega)$, $\mathcal{T}(j\omega)$
- These describe the response of the system in terms of disturbance rejection and tracking performance
- By shaping these, we can design a system with desired behaviour
- Complementarity represents an inherent tradeoff: tracking vs noise rejection
- Idea: Good tracking and noise rejection at low frequencies, bad at high

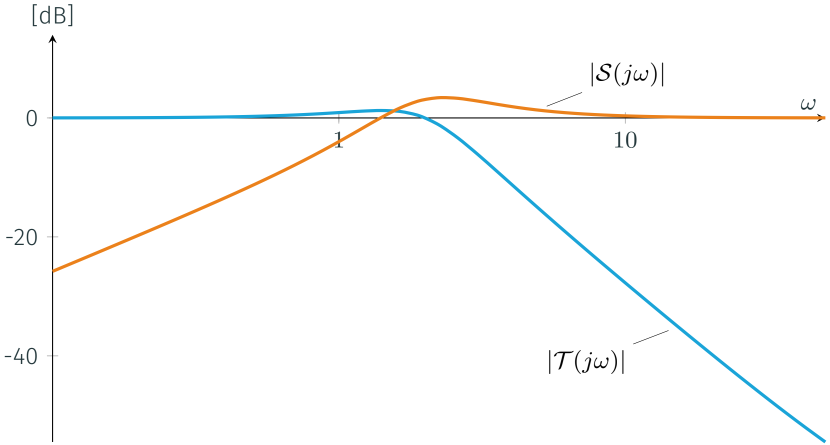
Complementary Sensitivity Function



Desired shape:

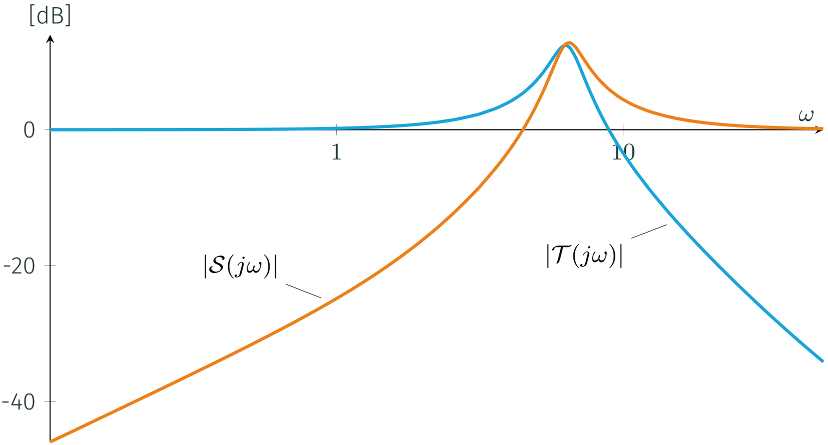
- Low-frequency gain of 0 dB
- Small resonance peak M_r at the resonant frequency ω_r
- Large bandwidth defined by the pass-band $[0, \omega_b]$, and the cutoff-frequency ω_b
- High roll-off after ω_b to make the system insensitive to measurement noise, and unmodeled dynamics

Example

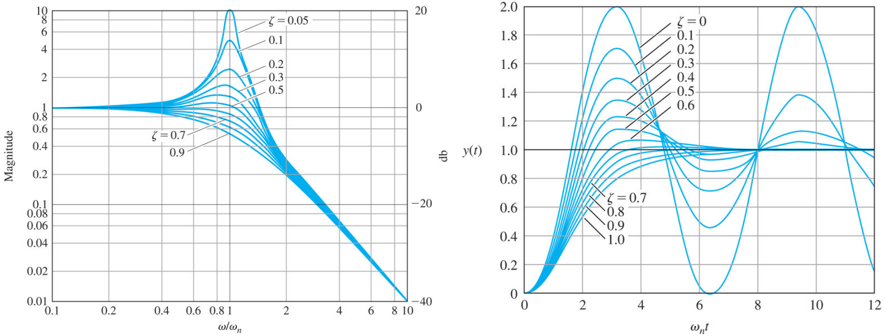


Low gain / high stability margin

Example

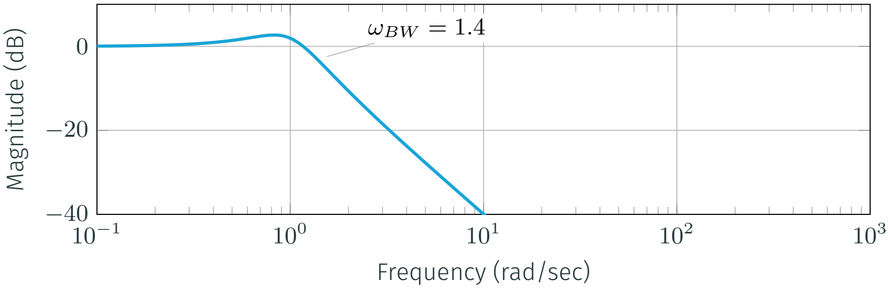


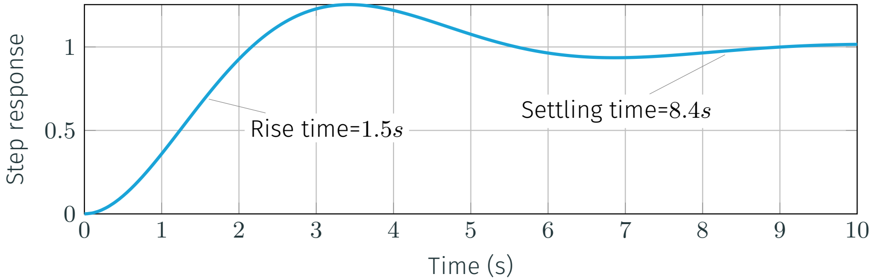
High gain / low stability margin

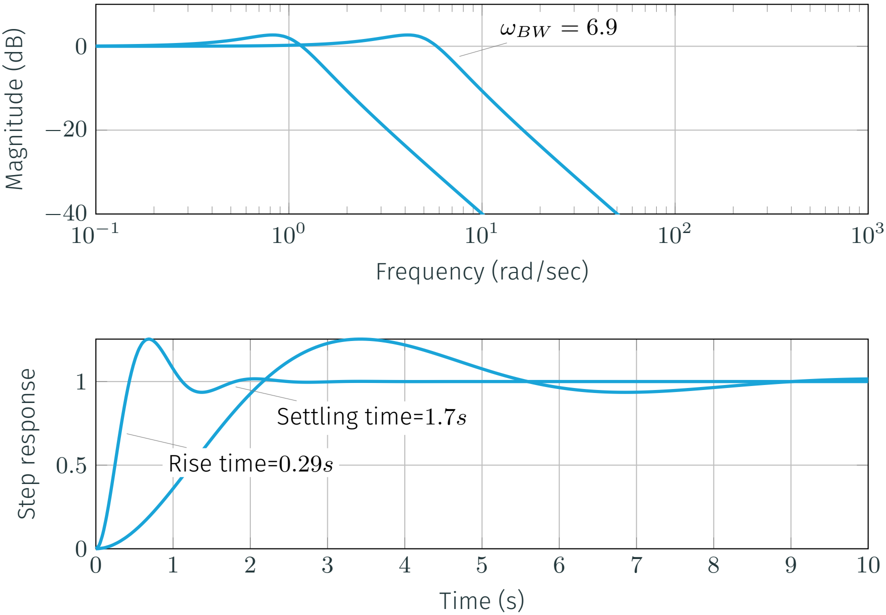


Magnitude of resonant peak related to the damping of the closed-loop system.

Bandwidth Defines Rise Time & Settling Time

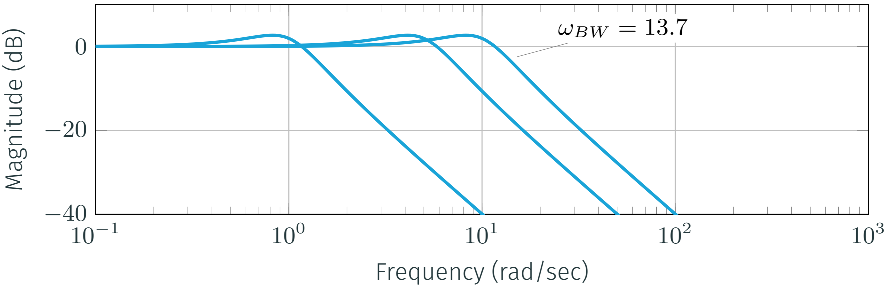


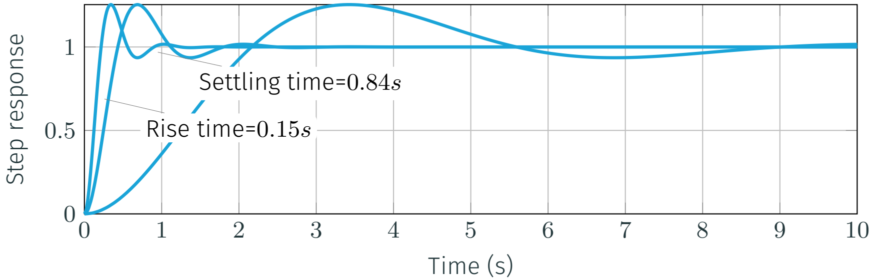




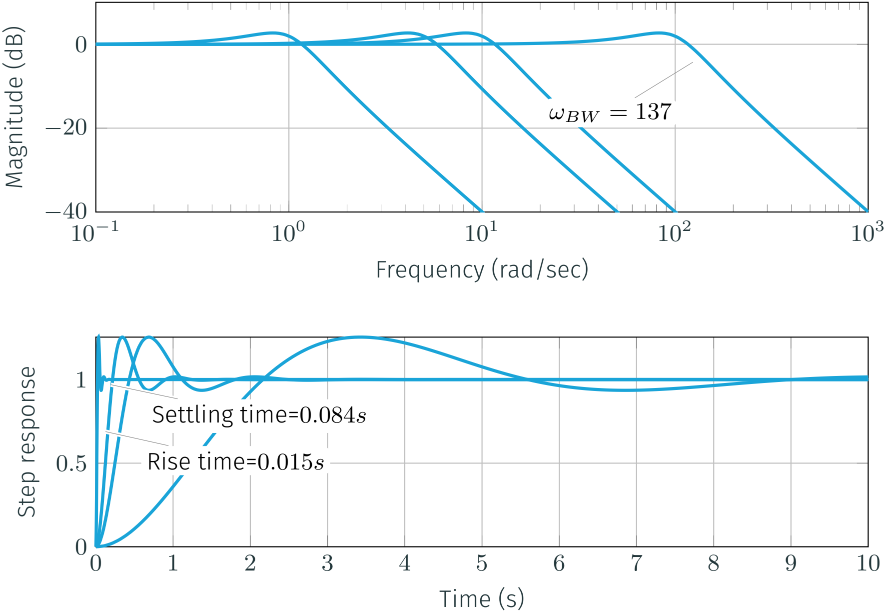
$\omega_{BW} = 6.9$
Magnitude (dB)
0
−20
−40
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)
Step response
1
0.5
0
0 1 2 3 4 5 6 7 8 9 10
Time (s)
Settling time=$1.7s$
Rise time=$0.29s$

Bandwidth Defines Rise Time & Settling Time





Bandwidth Defines Rise Time & Settling Time



Example 9.2

Loop Shaping

Open-Loop Properties $\Leftrightarrow$ Closed-Loop Properties

$$\mathcal{T}(j\omega) = \frac{K(j\omega)G(j\omega)}{1 + K(j\omega)G(j\omega)} = 1 - \frac{1}{1 + K(j\omega)G(j\omega)}$$

$\mathcal{T}(j\omega) = 1$ for small ω	$\leftrightarrow$	$K(j\omega)G(j\omega)$ large for small ω

$\rightarrow$ Integrator (pole at 0)

$\lvert\mathcal{T}(j\omega)\rvert \ll 0dB$ for large ω	$\leftrightarrow$	$\lvert K(j\omega)G(j\omega)\rvert \ll 0dB$ for large ω
Low resonance peak	$\leftrightarrow$	Large stability margins

$\rightarrow$ Resonance when $|1 + K(j\omega_r)G(j\omega_r)|$ is small

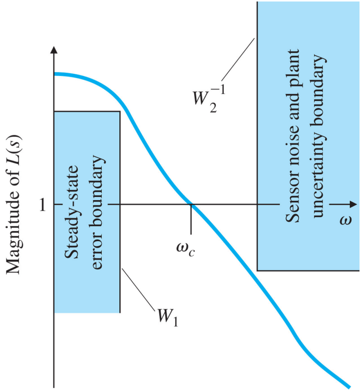
$\rightarrow$ $K(j\omega_r)G(j\omega_r) \approx -1$

Specified rise time/settling time	$\leftrightarrow$	Crossover frequency

$\rightarrow$ Open-loop crossover frequency $\approx$ Closed-loop bandwidth

Can describe good closed-loop behaviour via open-loop frequency response

Loopshaping Goals



(Note: $L(s) = K(s)G(s)$ is the ***Loop gain***)

Low-frequency slope (system type) and gain are chosen for steady-state error

High-frequency roll-off is determined by actuator/ sensor limitations and system bandwidth goals.

Closed-loop Bandwidth ≊ Crossover Frequency

The open-loop frequency response has been designed for

$$|KG(j\omega)| \gg 1 \text{ for } \omega \ll \omega_c$$

$$|KG(j\omega)| \ll 1 \text{ for } \omega \gg \omega_c$$

The closed-loop response is therefore

$$|\mathcal{T}(j\omega)| = \left|\frac{KG(j\omega)}{1+KG(j\omega)}\right| \approxeq \begin{cases} 1, & \omega \ll \omega_c \\ |KG|, & \omega \gg \omega_c \end{cases}$$

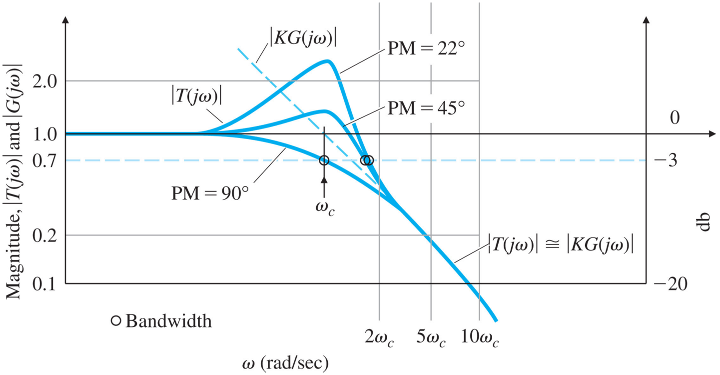
Around crossover, we have $|KG(j\omega)| \approx 1$ and $\mathcal{T}(j\omega)$ depends on the phase margin

$$KG(j\omega_c) = e^{j(\pi-\phi)} = -e^{-j\phi}$$

$$|\mathcal{T}(j\omega_c)| = \left|\frac{KG(j\omega_c)}{1+KG(j\omega_c)}\right| = \left|\frac{-e^{-j\phi}}{1-e^{-j\phi}}\right|$$

If $\phi = 90^\circ$, then $|\mathcal{T}(j\omega_c)| = 0.707 = -3\text{dB}$

Closed-loop Bandwidth ≈ Crossover Frequency

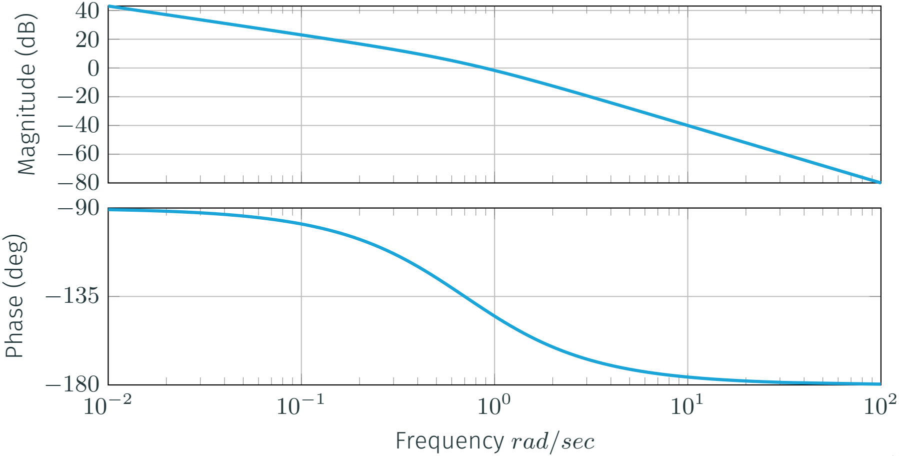


Closed loop bandwidth is within a factor of two of the crossover frequency

$$\omega_c \leq \omega_{BW} \leq 2\omega_c$$

Consider the prototype open-loop model

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$



Consider the prototype open-loop model

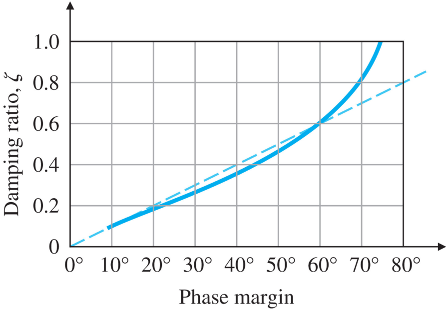
$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

With unity feedback, we get the closed-loop system

$$\mathcal{T}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$PM = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}\right]$$

$$\approx 100\zeta$$



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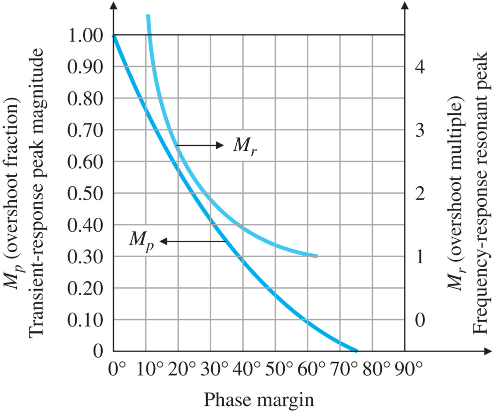


Damping ratio determines the step response overshoot, and the size of the resonant peak.

Loopshaping Goals

- KG large for small ω (Steady-state error)
- KG small for large ω (Modeling errors, etc)
- Crossover frequency chosen according to desired closed-loop bandwidth
- Good stability margins

Goal: Choose $K(s)$ to satisfy these requirements

Tools:

- Overall gain: Moves magnitude plot up and down
- Lead compensator
- Lag compensator

Lead Compensator

Phase Lead Compensator

Lead Compensator

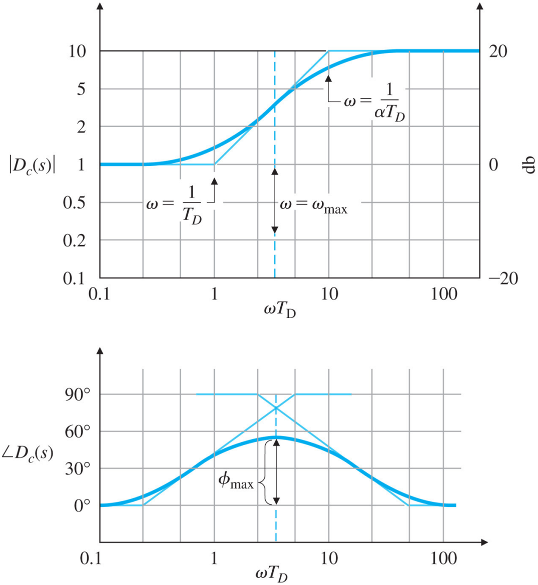
$$D_c(s) := \frac{T_D s + 1}{\alpha T_D s + 1} \qquad \alpha < 1$$

Within interval of interest

- Phase increased by $\phi_{\max}$
- Slope increased by 20dB/dec

Utility:

- Place near crossover frequency to increase phase



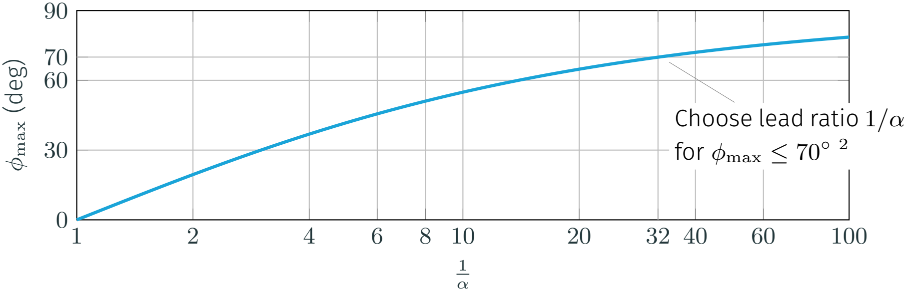
How much is the phase increased?

Max phase increase happens at the center of the pole and zero[1]

$$\omega_{\max} = \frac{1}{T_D\sqrt{\alpha}} \qquad \log\omega_{\max} = \frac{1}{2}\left(\log\frac{1}{T_D} + \log\frac{1}{\alpha T_D}\right)$$

The amount of phase lead at this point is

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha} \qquad \alpha = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$



[1] See Problem 6.44

[2] Or gain at high frequencies may be too much, and multiple lead compensators should be used.

Example

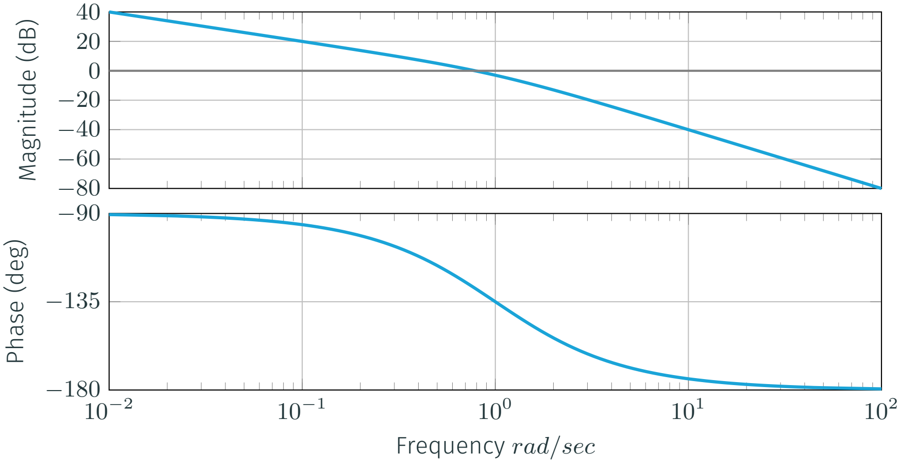
Consider the following system

$$G(s) = \frac{1}{s(s+1)}$$

Requirements:

- Steady-state error less than 0.1 in response to a ramp reference
- Overshoot of less than $M_P \leq 25\%$

Gain margin $= \infty$ Phase margin $= 52^\circ$



Example

Steady-state error less than 0.1 in response to a ramp reference

This is a Type 1 system:

$\rightarrow$ Error with respect to a ramp input is $\frac{1}{\gamma}$

$$\gamma = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{1}{s+1} = 1$$

Steady-state error in response to a ramp is $e_{ss} = 1$.

Example

Steady-state error less than 0.1 in response to a ramp reference

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$$\gamma = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{1}{s+1} = 1$$

Steady-state error in response to a ramp is $e_{ss} = 1$.

Try the simplest controller to improve this: Proportional gain K

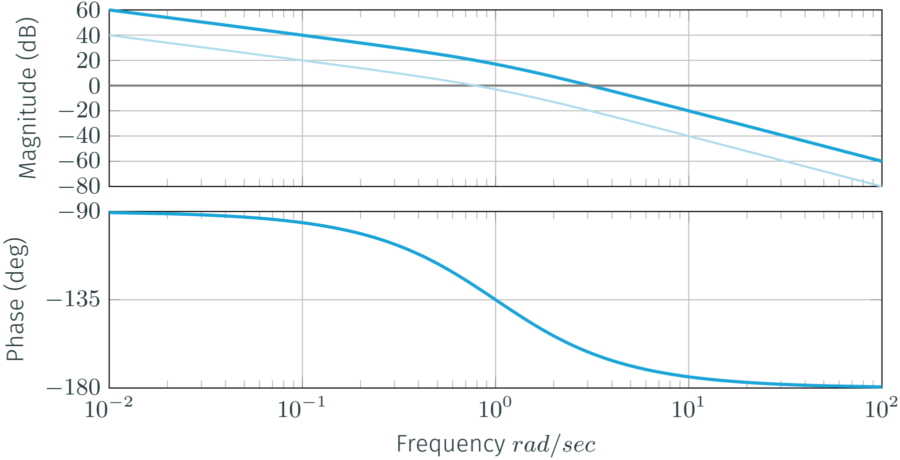
Loop gain is now $L(s) = KG(s) = \frac{K}{s(s+1)}$.

$$\gamma = \lim_{s \to 0} sKG(s) = \lim_{s \to 0} \frac{K}{s+1} = K$$

Steady-state error in response to a ramp is $e_{ss} = \frac{1}{K}$.

$\rightarrow$ Choose $K = 10$

Gain margin $= \infty$ Phase margin $= 18^\circ$



Example

Overshoot of less than $M_P \leq 25\%$

From Slide 16 we see that a phase margin of $45°$ will do

$\rightarrow$ Add a phase lead compensator

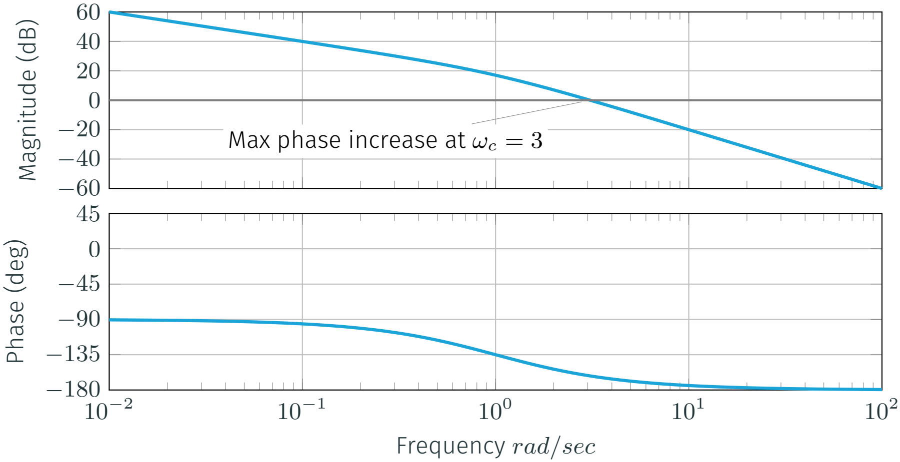
Current phase margin is $\approx 20°$ $\rightarrow$ Requires an increase of $25°$

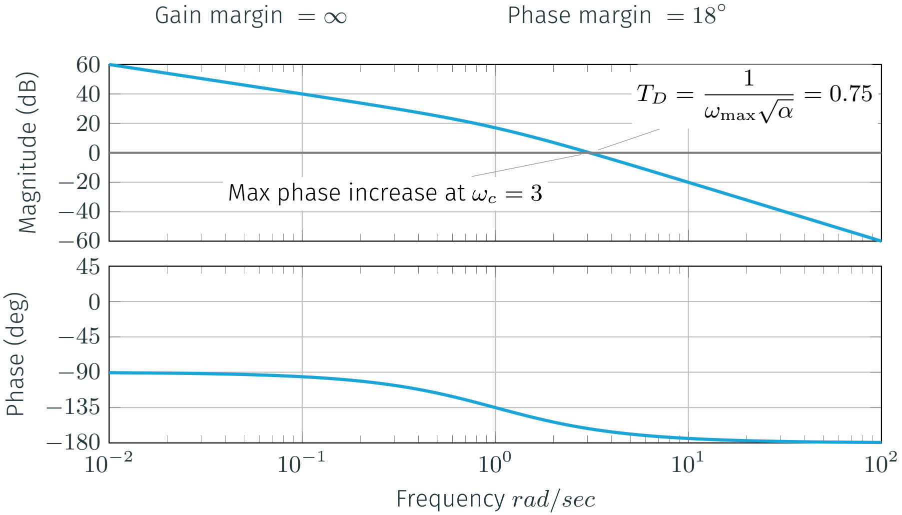
Lead compensator also increases gain $\rightarrow$ Increases crossover frequency

Increase phase by $\approx 40°$ to compensate

Slide 19 shows $\alpha = 1/5$ will increase phase by $\approx 40°$

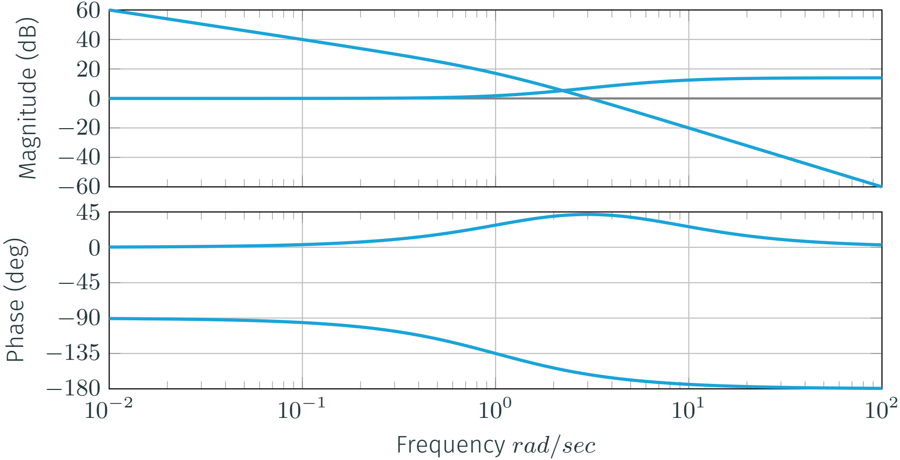
Gain margin $= \infty$ Phase margin $= 18^\circ$




Gain margin $= \infty$
Phase margin $= 18^\circ$
$T_D = \dfrac{1}{\omega_{\max}\sqrt{\alpha}} = 0.75$
Max phase increase at $\omega_c = 3$
Magnitude (dB)
60
40
20
0
-20
-40
-60
Phase (deg)
45
0
-45
-90
-135
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency rad/sec

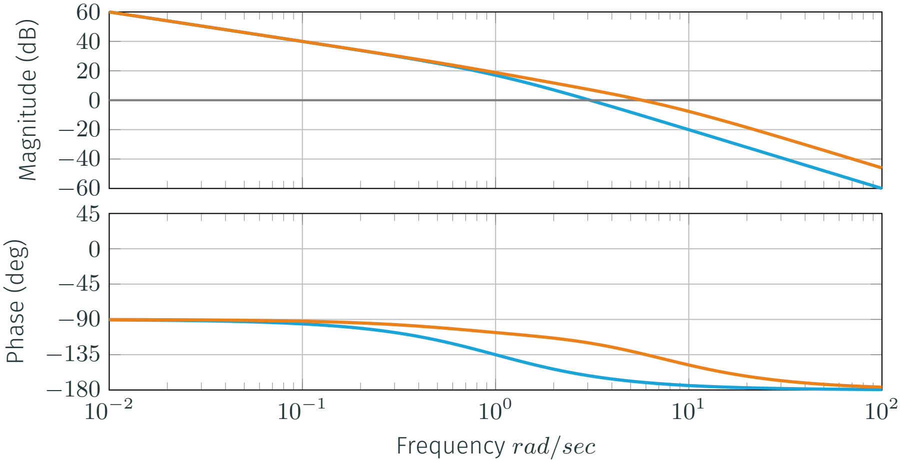
Example

Gain margin $= \infty$ Phase margin $= 18^\circ$

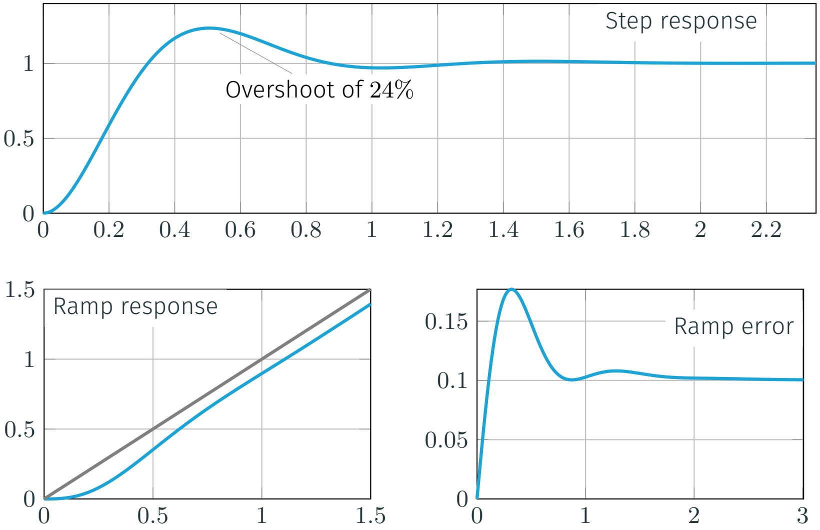


Example

Gain margin $= \infty$ Phase margin $= 46^\circ$



Example - Time Domain Result $G(s) = \frac{1}{s(s+1)}$ $K(s) = 10\frac{0.75s+1}{0.15s+1}$



Lead Design Summary

Generally three criteria

1.	Crossover frequency	$\leftarrow$	Bandwidth, rise time and settling time
2.	Phase margin	$\leftarrow$	Damping coefficient ζ and overshoot M_p
3.	Low-frequency gain	$\leftarrow$	Steady-state error

Design procedure

1. Choose system type and controller gain K such that
   - steady-state gain targets are met
   - open-loop crossover frequency is a factor of two below the desired closed-loop bandwidth
2. Determine the increase in phase margin required (add about 10° to compensate for bandwidth increase) and choose α to give the desired increase.
3. Choose $\omega_{\max}$ to be the crossover frequency, and set $T_D = \frac{1}{\omega_{\max}\sqrt{\alpha}}$

Note that this procedure may require customization for any particular system.

Main idea: A low slope at crossover provides a good phase margin.

e.g., -20dB/dec gives a phase margin of about $90°$

Slope must be constant for a decade around the crossover frequency for approximation to hold. Equivalent to choosing $1/\alpha = \sqrt{5} \approx 3$.

$$D_c(s) := \frac{3s + \omega_c}{s/3 + \omega_c}$$

Ignore the phase plot, and work only with the magnitude plot.

Bode Gain-Phase Theorem

For any stable minimum-phase system (i.e., one with no RHP zeros or poles), the phase of $G(j\omega)$ is uniquely related to the magnitude of $G(j\omega)$

$$\angle G(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{dM}{du}\right)W(u)du \qquad \text{(in radians)}$$

where

- $M =$ log Magnitude = $\ln|G(j\omega)|$
- $u =$ normalized frequency $= \ln(\omega/\omega_0)$
- $W(u) =$ weighting function $= \ln(coth|u|/2)$

Gain-Phase Relationship

Bode Gain-Phase Theorem (Simple form)

$$\angle G(j\omega) \approxeq n \times 90^\circ$$

where n is the slope of $|G(j\omega)|$ in units of decade of amplitude per decade of frequency.

If the crossover frequency is ω_0, i.e., the gain is $|K(j\omega_0)G(j\omega_0)| = 1$, then

- $\angle G(j\omega_0) \approx -90^\circ$ if $n = -1$ (-20dB / dec)
- $\angle G(j\omega_0) \approx -180^\circ$ if $n = -2$ (-40dB / dec)

Main idea: A low slope at crossover provides a good phase margin.

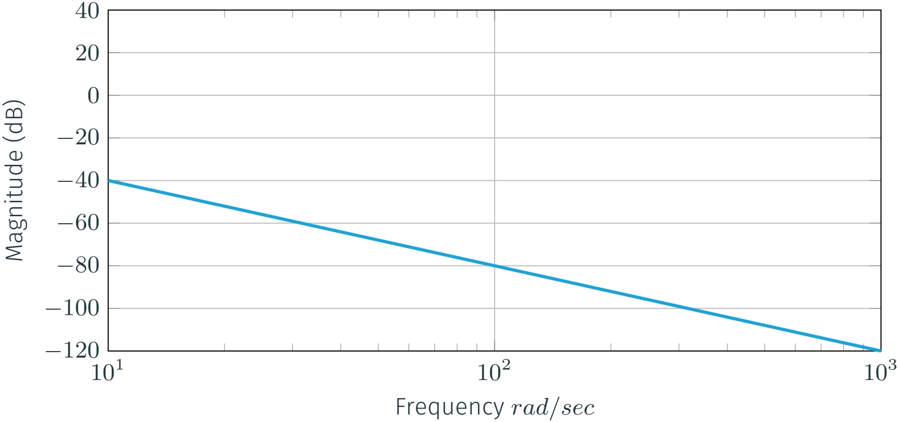
e.g., -20dB/dec gives a phase margin of about 90°

[2] Slope must be constant for a decade around the crossover frequency for approximation

Simple Example

$$G(s) = \frac{1}{s^2}$$

Design a lead compensator for the system providing zero steady-state error in response to a ramp input, around 60° phase margin and a bandwidth of at least $100r/s$.



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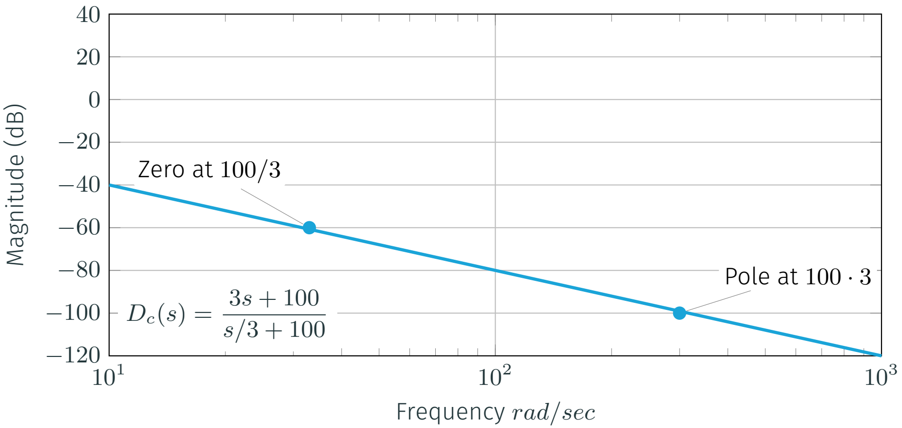

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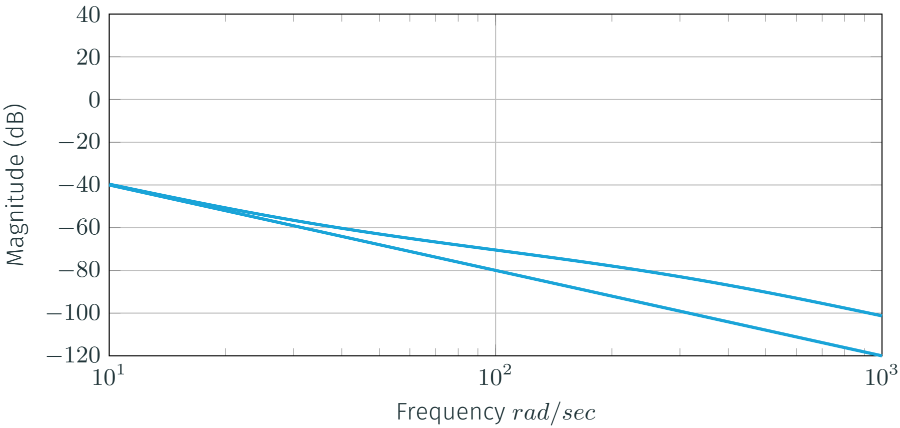
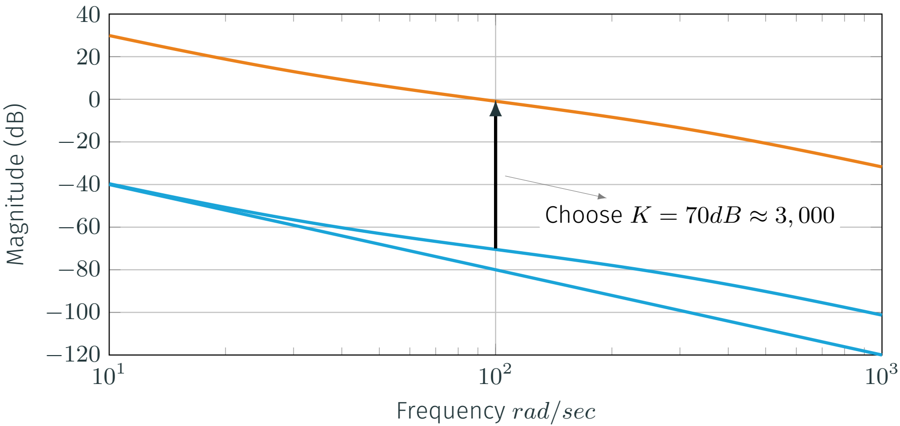


$$G(s) = \frac{1}{s^2}$$

Design a lead compensator for the system providing zero steady-state error in response to a ramp input, around $60°$ phase margin and a bandwidth of at least $100 r/s$.



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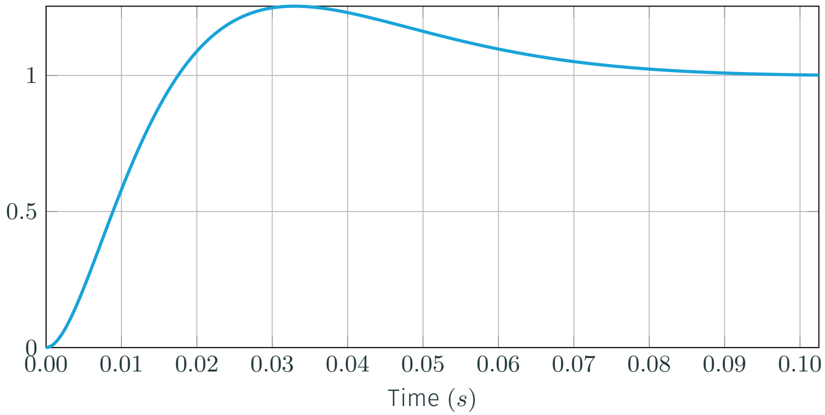
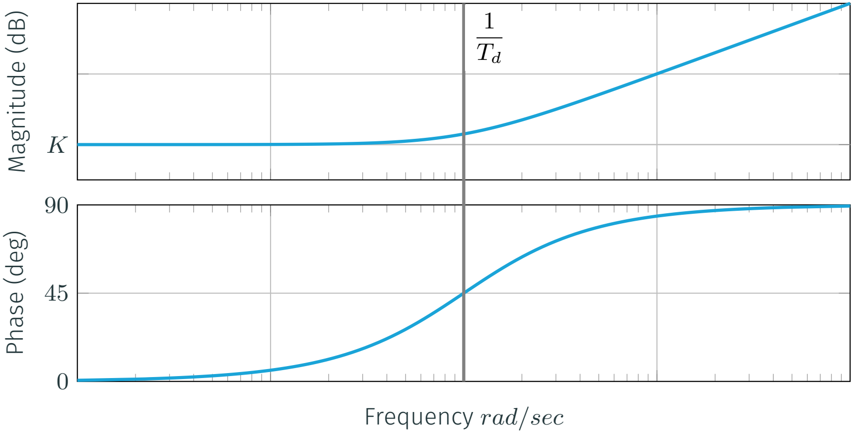


Interpretation of PD Controller

Consider the PD controller:

$$K(s) = K(1 + T_D s)$$

This is a lead compensator with the pole at $s = -\infty$, or $\alpha = 0$.



$$G(s) = 0.05\frac{80 - s}{s(s + 2.05)}$$

Design a PD controller such that:

- Steady-state error to step input is zero
- Track ramp with steady-state error less than 0.05
- Closed-loop step response with time-constant less than 0.07s
- Phase margin greater than 60°

- Steady-state error to step input is zero
- Track ramp with steady-state error less than 0.05

Consider a proportional controller: $K(s) = K$

- Steady-state error to step input is zero
- Track ramp with steady-state error less than 0.05

Consider a proportional controller: $K(s) = K$

$$K(s)G(s) = K \cdot 0.05 \frac{80 - s}{s(s + 2.05)}$$

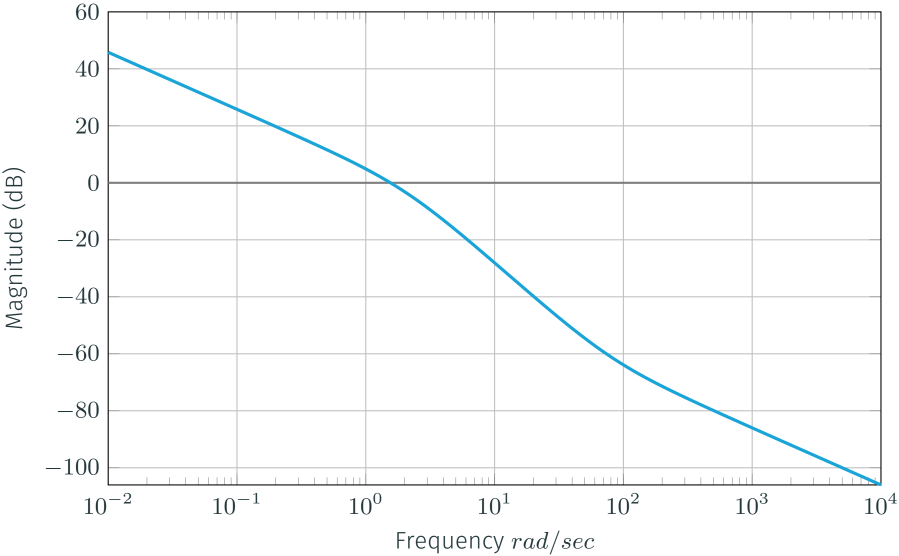
Type 1 system

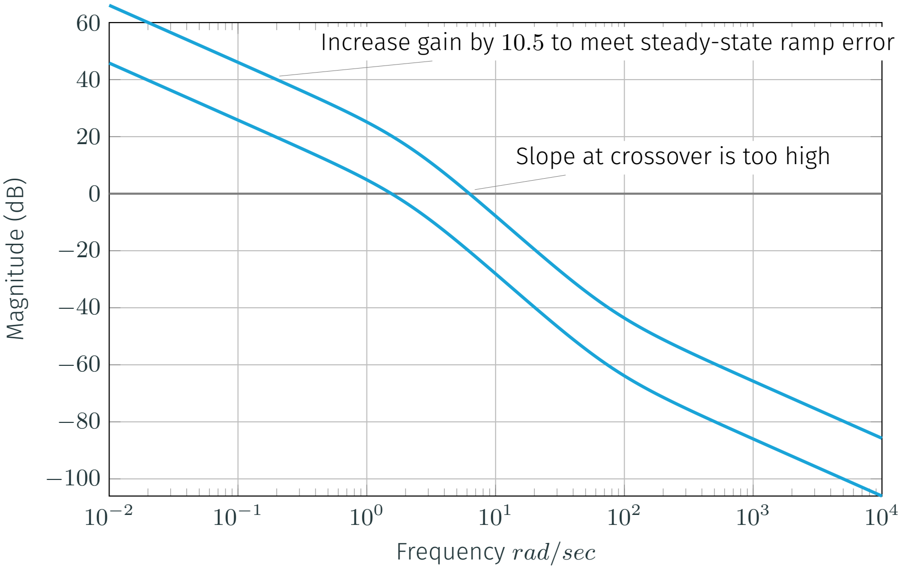
- Zero steady-state error to a step
- Steady-state error to a ramp reference $r(t) = t$ is $1/\gamma$

$$\gamma := \lim_{s \to 0} s^q K(s)G(s) = K \frac{B(0)}{A(0)} = K \cdot 0.05 \frac{80}{2.05} = K \cdot 1.9$$

Error $= 1/(K \cdot 1.9) \leq 0.05$
Therefore, the gain $K \geq 1/(0.05 \cdot 1.9) = 10.5$

Frequency Response




Increase gain by 10.5 to meet steady-state ramp error
Slope at crossover is too high
Magnitude (dB)
60
40
20
0
−20
−40
−60
−80
−100
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency rad/sec

Add Lead Compensator

Goal: Improve phase margin

Add derivative term (Lead compensator)

$$K(s) = 10.5(1 + T_D s)$$

How to choose T_D?

- Sets bandwidth of the system
- Roughly sets the time constant of the closed-loop step response

Bandwidth and Time Constant

Assume: Phase margin of about 90° at a crossover frequency of ω_c
For frequencies near ω_c, the open-loop gain is approximately:

$$K(j\omega)G(j\omega) \approx \frac{\omega_c}{j\omega}$$

The step response is approximately:

$$Y(s) = \frac{K(s)G(s)}{1 + K(s)G(s)} \cdot \frac{1}{s} \approx \frac{\frac{\omega_c}{s}}{1 + \frac{\omega_c}{s}} \cdot \frac{1}{s} = \frac{\omega_c}{s + \omega_c} \cdot \frac{1}{s} = \frac{-1}{s + \omega_c} + \frac{1}{s}$$

Gives the time response:

$$y(t) = 1 - e^{-\omega_c t}$$

The system time constant is approximately $1/\omega_c$

Add Lead Compensator

Goal: Improve phase margin

Add derivative term (Lead compensator)

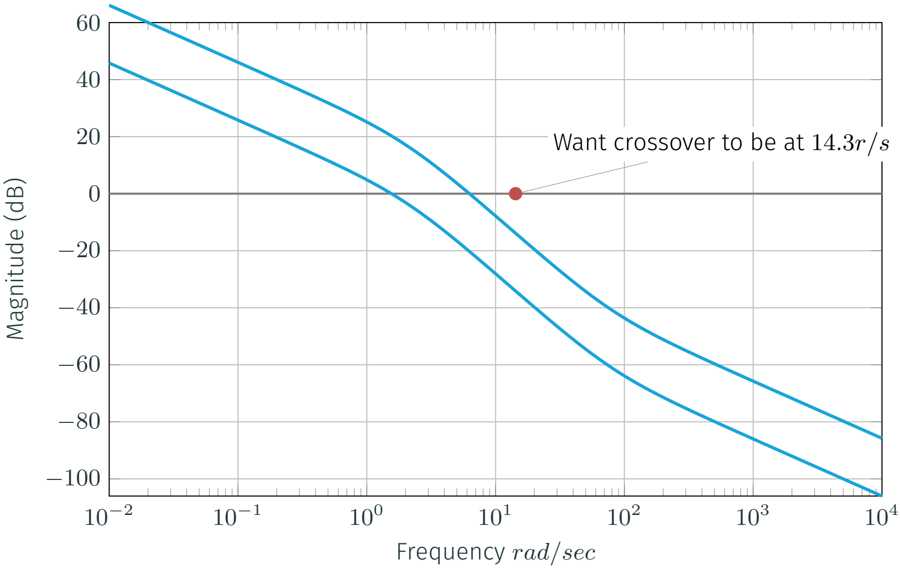
$$K(s) = 10.5(1 + T_D s)$$

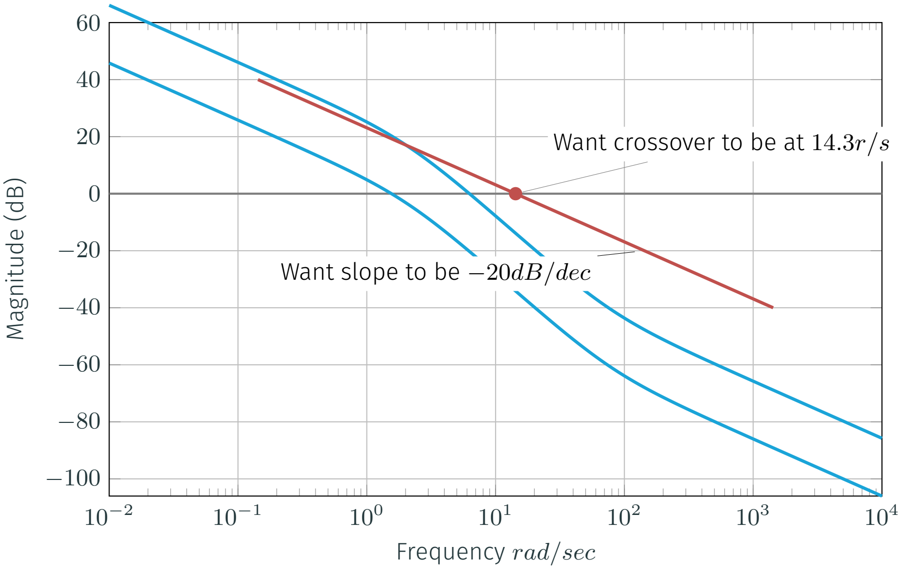
How to choose T_D?

- Sets bandwidth of the system
- Roughly sets the time constant of the closed-loop step response

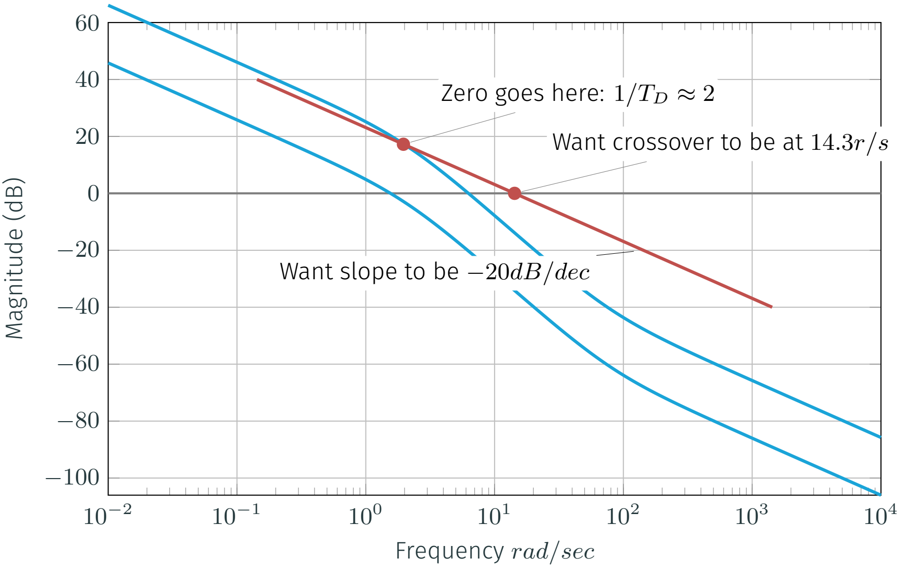
Goal: Time constant less than 0.07s

Choose $\omega_c \geq 1/0.07 = 14.3$

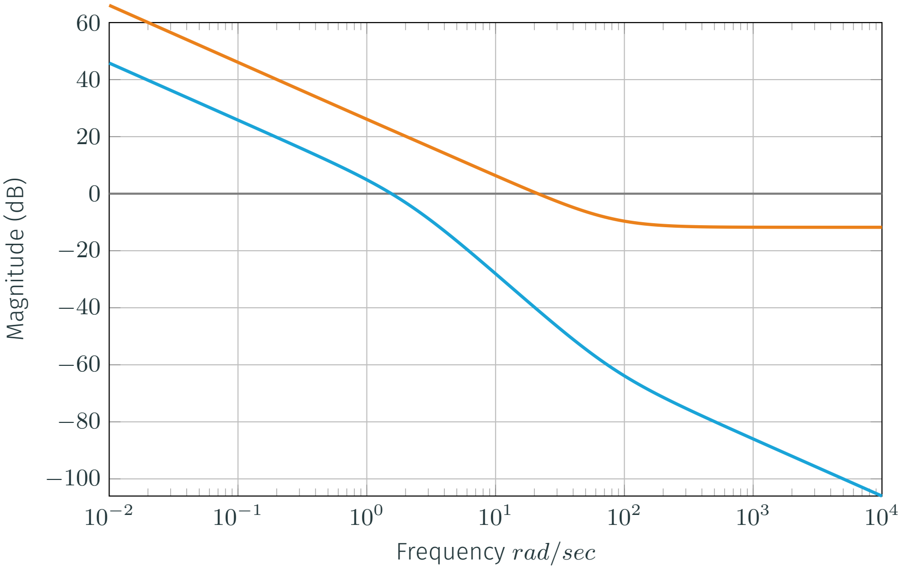

Want crossover to be at $14.3r/s$
Magnitude (dB)
60
40
20
0
−20
−40
−60
−80
−100
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency rad/sec


60
40
20
0
−20
−40
−60
−80
−100
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
Frequency rad/sec
Want crossover to be at $14.3r/s$
Want slope to be $-20dB/dec$

Frequency Response



Frequency Response



Example - PD Controller

$$G(s) = 0.05\frac{80 - s}{s(s + 2.05)}$$

Design a PD controller such that:

- Steady-state error to step input is zero
- Track ramp with steady-state error less than 0.05
- Closed-loop step response with time-constant less than 0.07s
- Phase margin greater than 60°

Our final controller is:

$$K(s) = 10.5 \cdot (1 + s/2)$$

Example 9.10

Lead Compensator

Benefit

- Increase the phase near the crossover frequency

Downside

- Increases the gain at high-frequencies
  - $\rightarrow$ Increases sensitivity to noise and unmodeled dynamics

Lag Compensator

Lead Compensator

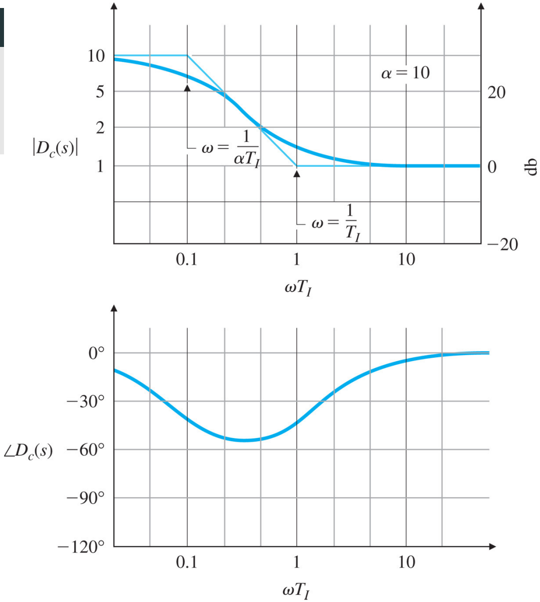
$$D_c(s) := \alpha \frac{T_I s + 1}{\alpha T_I s + 1} \qquad \alpha > 1$$

Within interval of interest

- Phase decreased by up to $90°$
- Gain increased below frequency $1/T_I$

Utility:

- Increase gain at low frequencies to reduce steady-state errors



Phase Lag Compensator

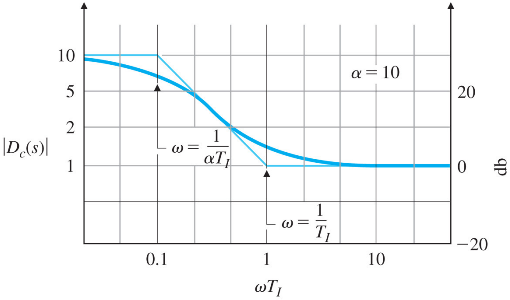
Goal

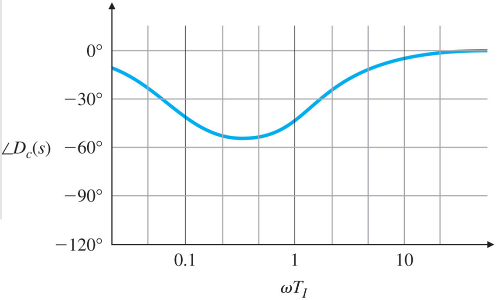
Increase low-frequency gain, without impacting transient behaviour

Idea

- Set break frequency $\frac{1}{T_I}$ below the crossover frequency, to not impact transient behaviours
- Choose α to give desired steady-state behaviour

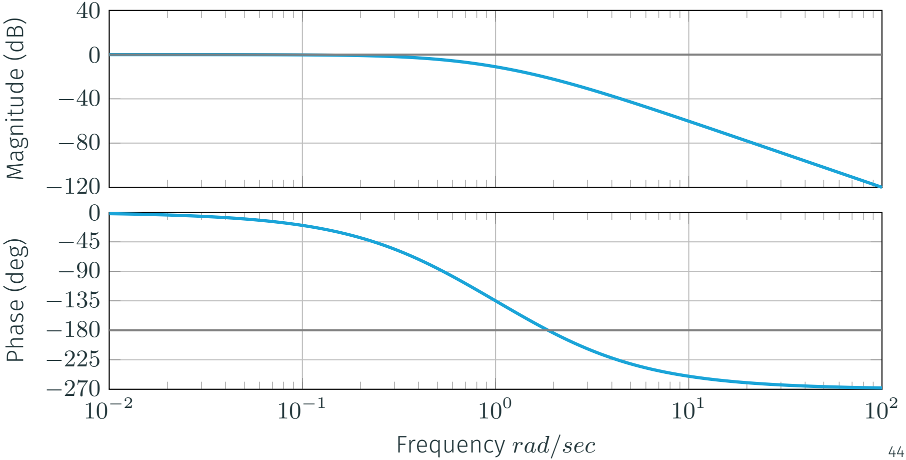
$$\lim_{s \to 0} \alpha \frac{T_I s + 1}{\alpha T_I s + 1} = \alpha$$





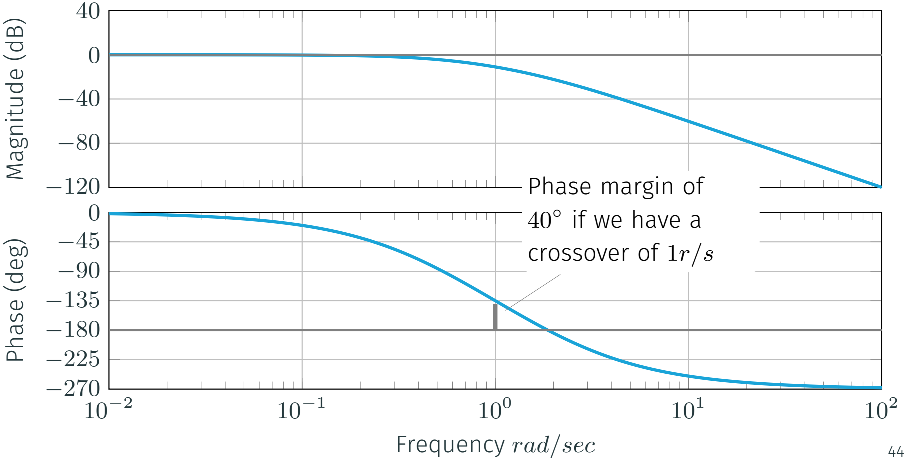
$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

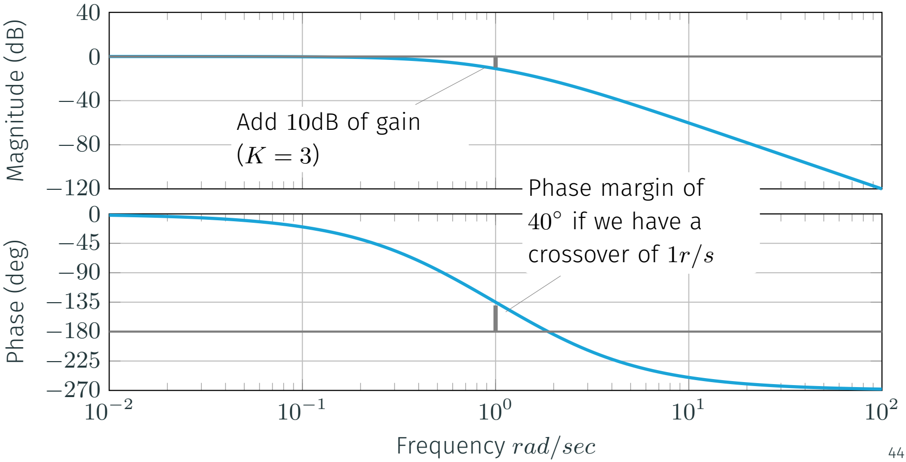
Design a lag compensator to achieve a phase margin of at least 40° and a steady-state error with respect to a step input better than 10%.



Example

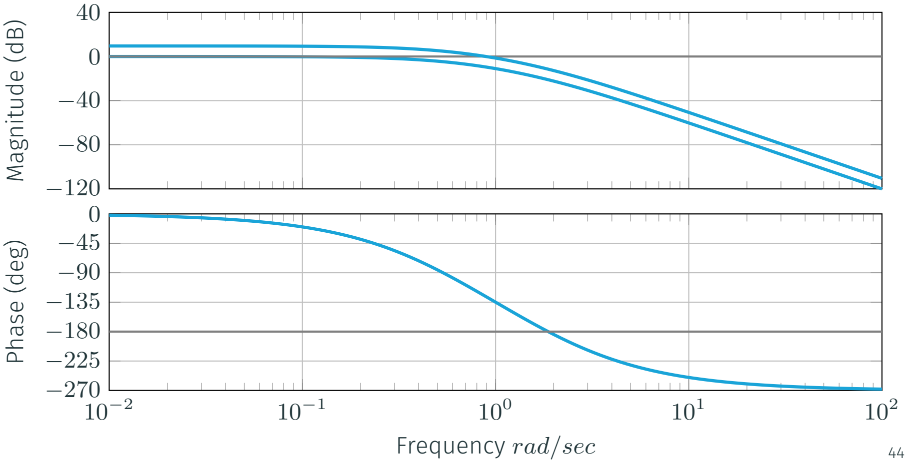
$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

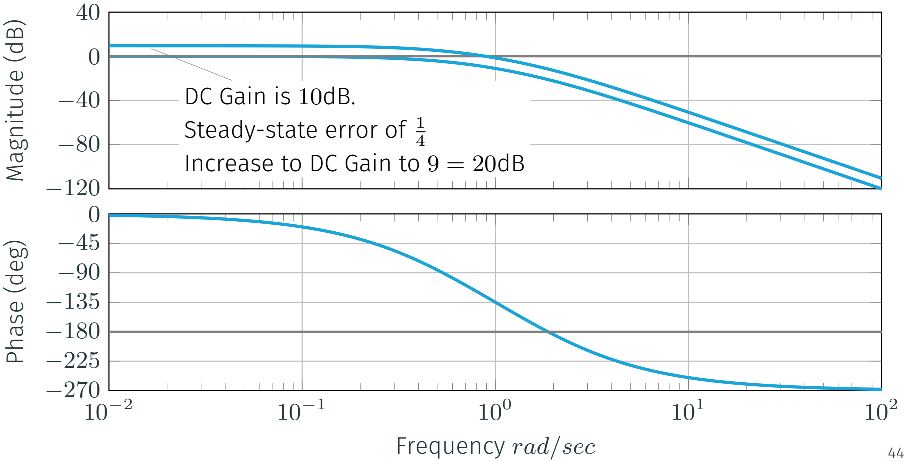
Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



Example

$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

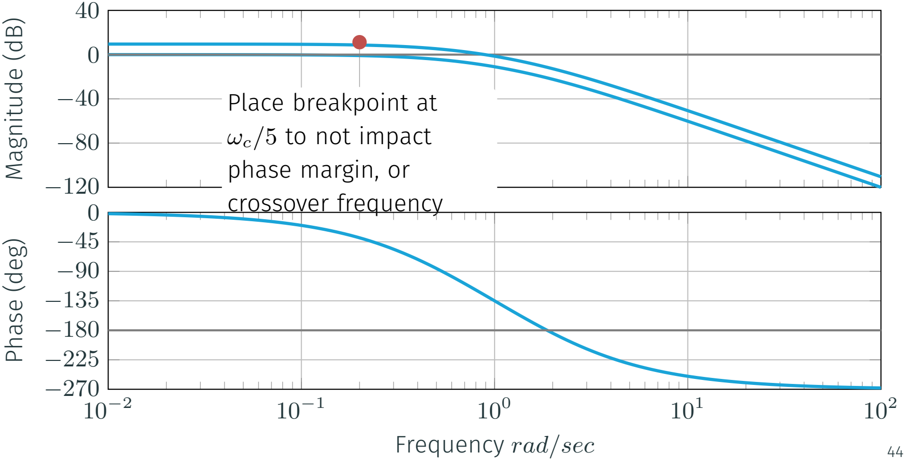
Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



Example

$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

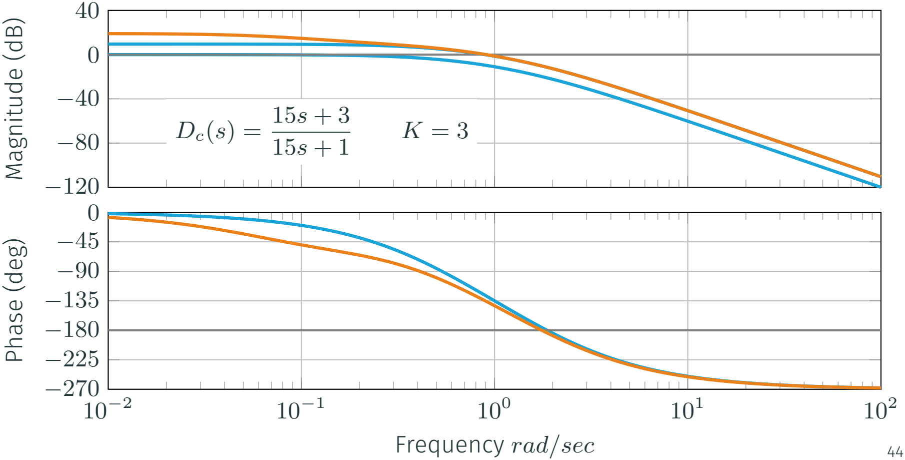
Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



Example

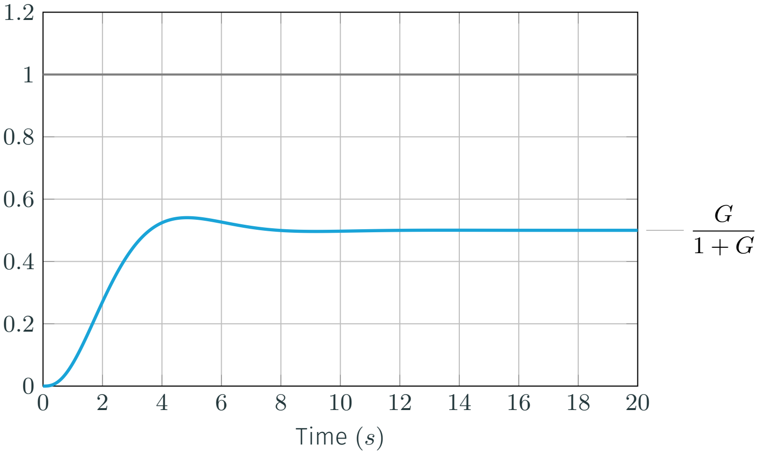
$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



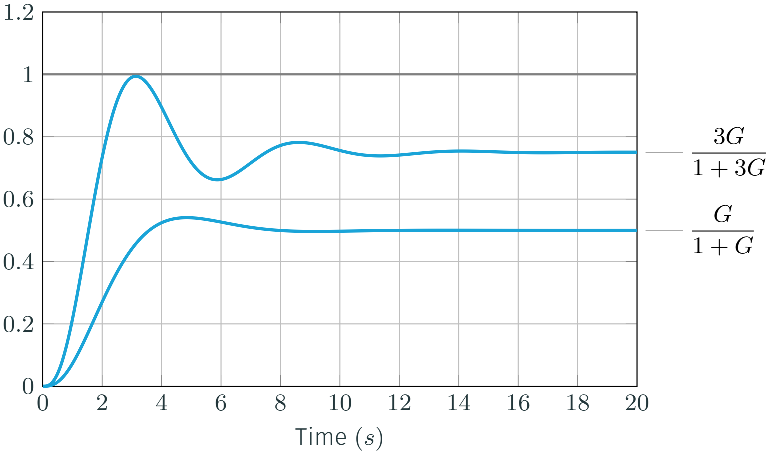
$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



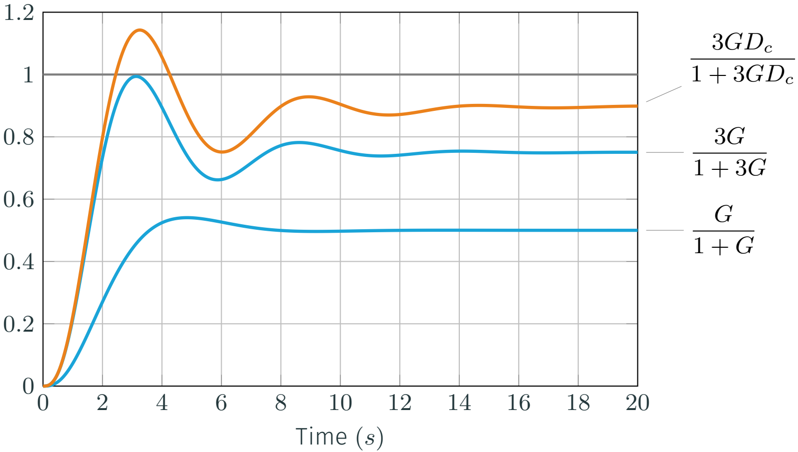
$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.



$$G(s) = \frac{1}{(\frac{1}{0.5}s+1)(s+1)(\frac{1}{2}s+1)}$$

Design a lag compensator to achieve a phase margin of at least $40°$ and a steady-state error with respect to a step input better than 10%.

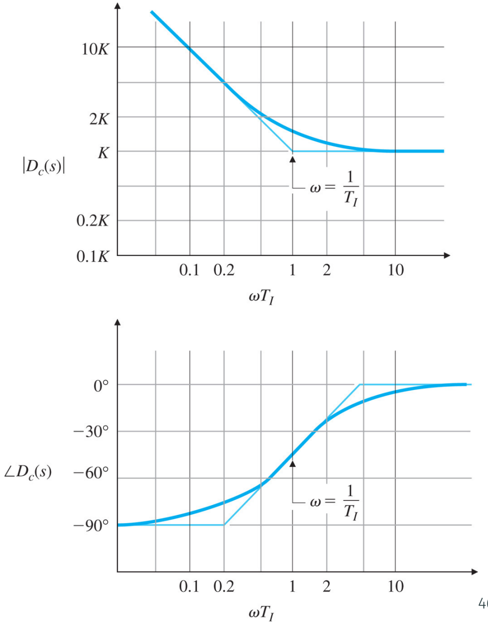


Relation to PI Controller

What happens when $\alpha \to \infty$?

$$
\begin{aligned}
KD_c(s) &= K\alpha\frac{T_I s + 1}{\alpha T_I s + 1} \\
&= K\frac{T_I s + 1}{T_I s + \frac{1}{\alpha}} \\
&= K\left(1 + \frac{1}{T_I s}\right)
\end{aligned}
$$

We see that a lag compensator is a PI controller with $\alpha = \infty$



Example

$$G(s) = 0.05\frac{80 - s}{s + 2.05}$$

Specifications:

- Zero steady-state error to step command
- Phase margin greater than 60°
- Closed-loop time constant of $1/\omega_c = 0.07$s ($\omega_c = 14.3$rad/s)

Example 9.11

Lead/Lag Compensator

Lead/Lag Compensators

Lead compensator	Adds phase at crossover frequency to improve margins
	Impacts frequencies ***above*** the breakpoint
Lag compensator	Adds gain at low frequency to improve steady-state response
	Impacts frequencies ***below*** the breakpoint

We are free to use lead and lag filters in combination, without them impacting each other, often called lead-lag filters.

Lead $\rightarrow$ PD controller

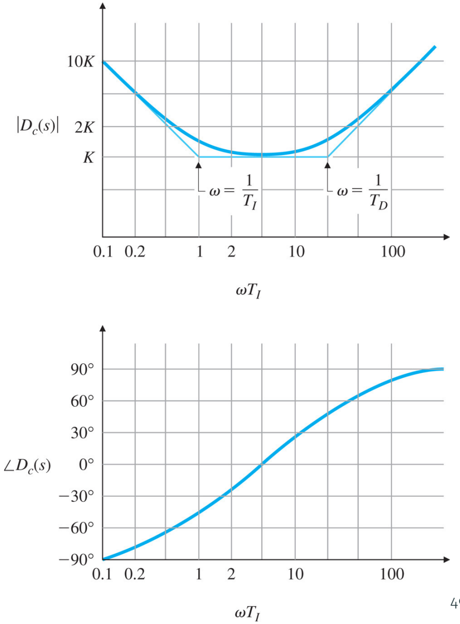
Lag $\rightarrow$ PID controller

A PID controller is a lead/lag filter

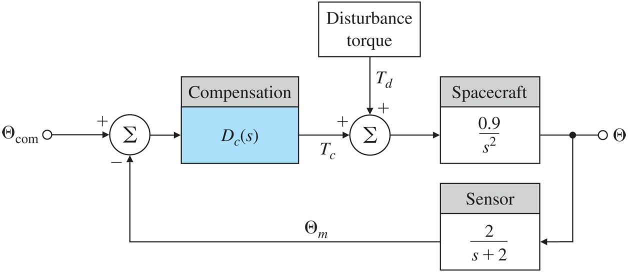
$$D_c(s) = K\left(T_D s + 1\right)\left(1 + \frac{1}{T_I s}\right)$$

PID Lead/Lag Filter

Frequency response of PID compensator for $\frac{T_I}{T_D} = 20$



Satellite Stabilization Problem

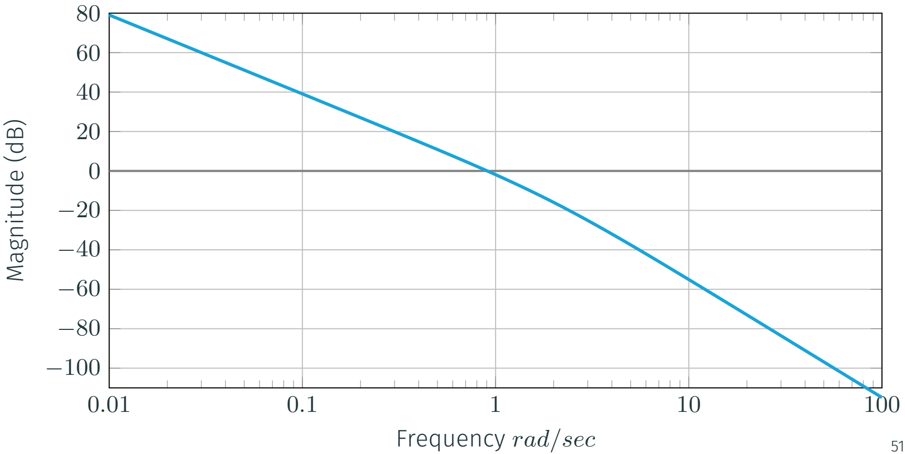


Design a PID controller for

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible

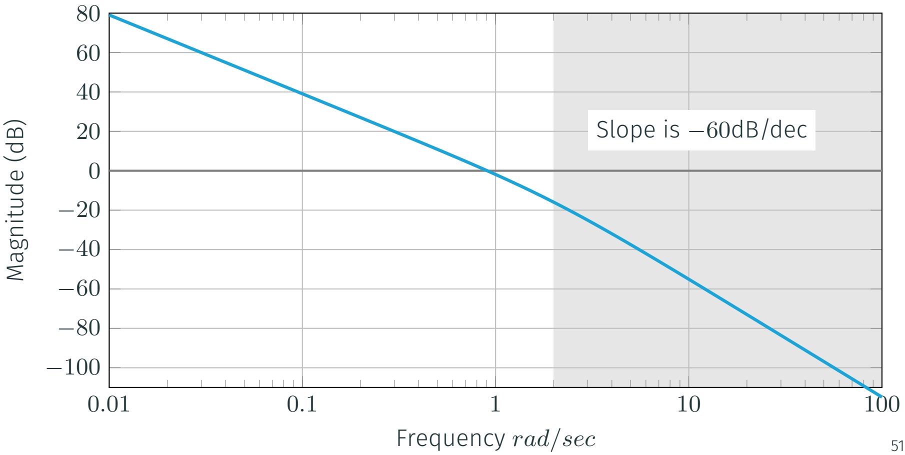
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately 60°
- As high a bandwidth as possible



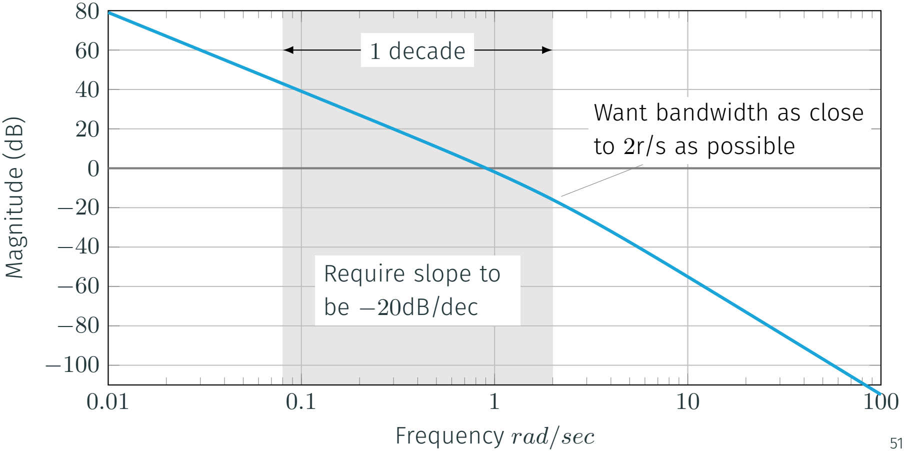
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately 60°
- As high a bandwidth as possible



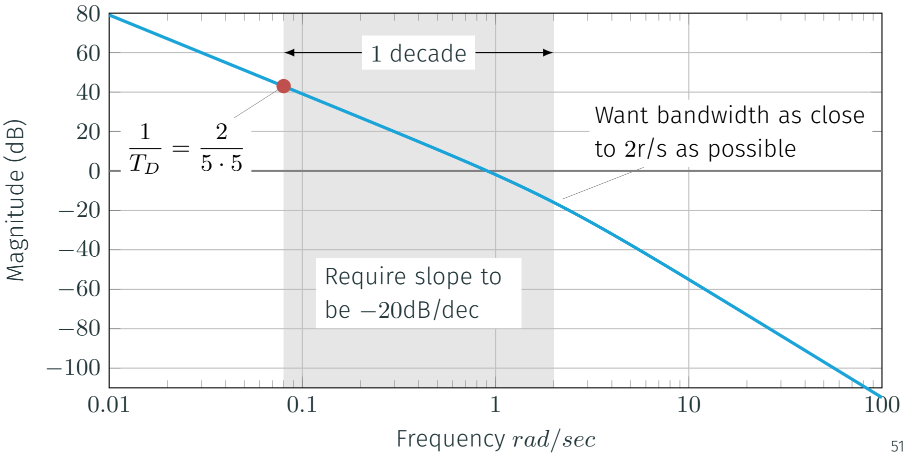
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately 60°
- As high a bandwidth as possible



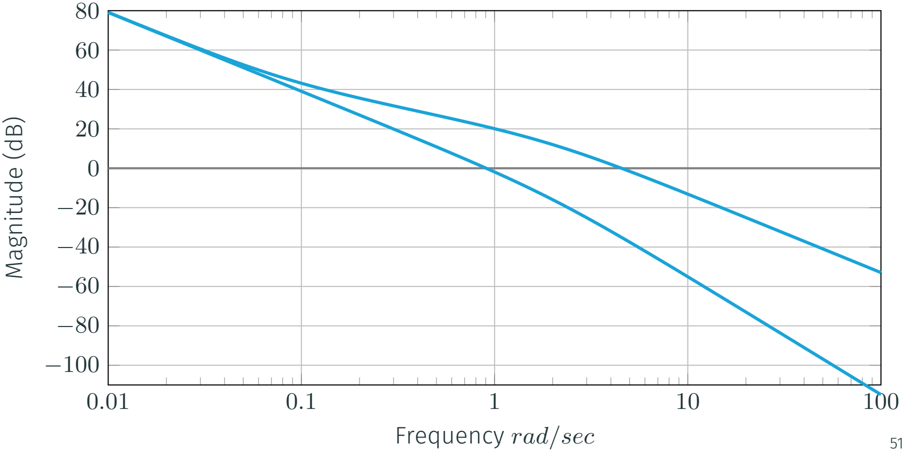
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible



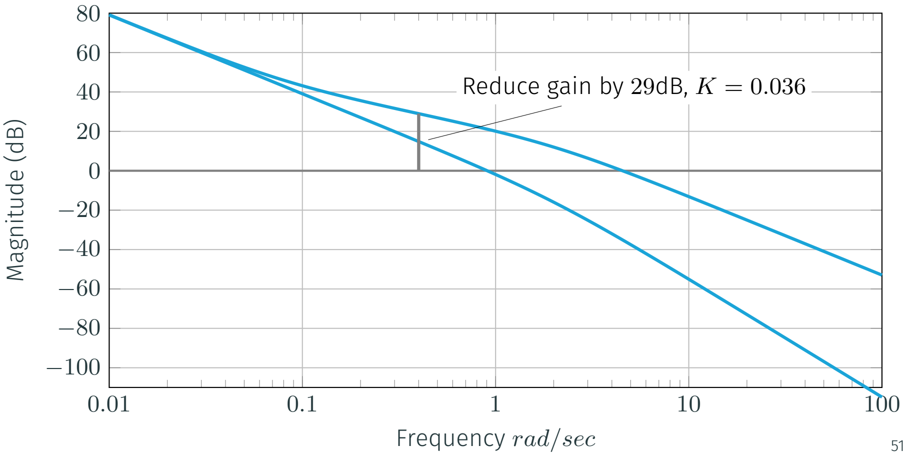
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible



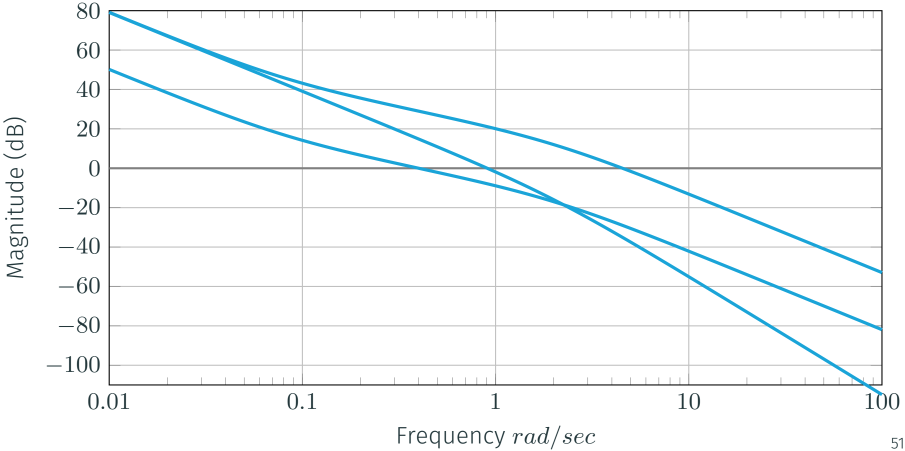
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible



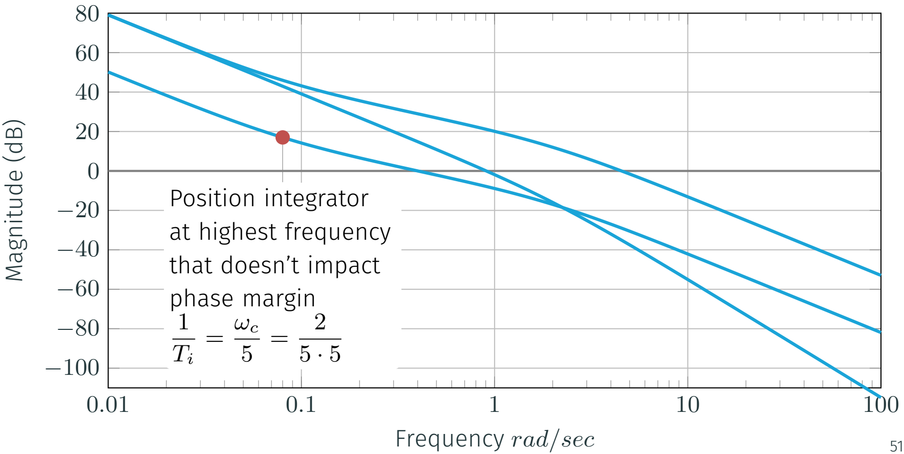
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible



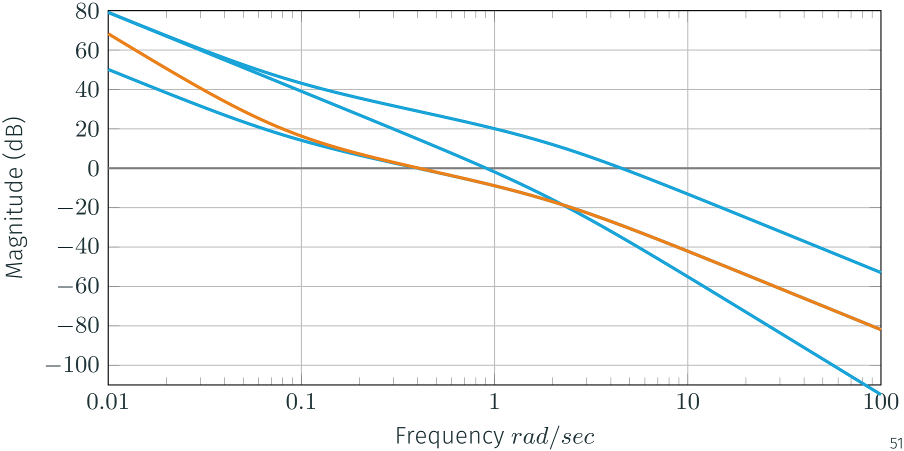
Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible

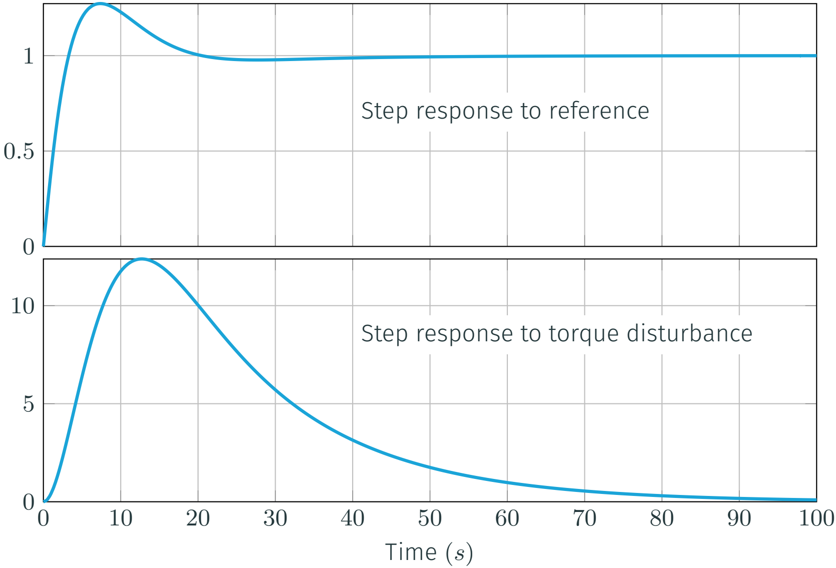


Example

- Zero steady-state error in response to a step disturbance torque
- A phase margin of approximately $60°$
- As high a bandwidth as possible



Time Response



Summary - Loop Shaping

Idea Can relate the shape of the frequency response of the open-loop system to the closed-loop sensitivity and complementary sensitivity functions

Goal

- KG large for small ω (Steady-state error)
- KG small for large ω (Modeling errors, etc)
- Crossover frequency chosen according to desired closed-loop bandwidth
- Good stability margins / slope of KG equal to -20dB/dec at crossover

Lead compensator

- Increase slope by 20dB/dec in frequency range
- Use to increase slope / phase near crossover frequency
- PD controller is a lead compensator

Lag compensator

- Use to increase gain at low frequencies
- Decreases slope by 20dB/dec / decreases phase in frequency range
- PI controller is a lag compensator

A Real System Design

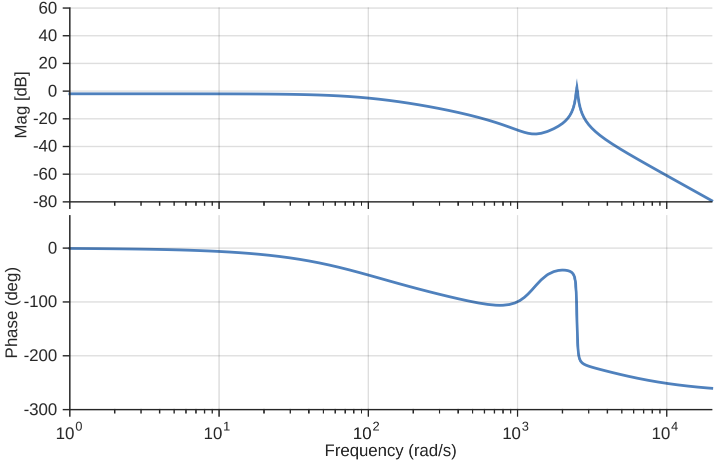
Atomic Force Microscope

$$G(s) = 8.88 \cdot 10^8 \frac{s^2 + 780s + 1.69 \cdot 10^6}{(s+3000)(s+1000)(s+100)(s^2+50s+6.25 \cdot 10^6)}$$

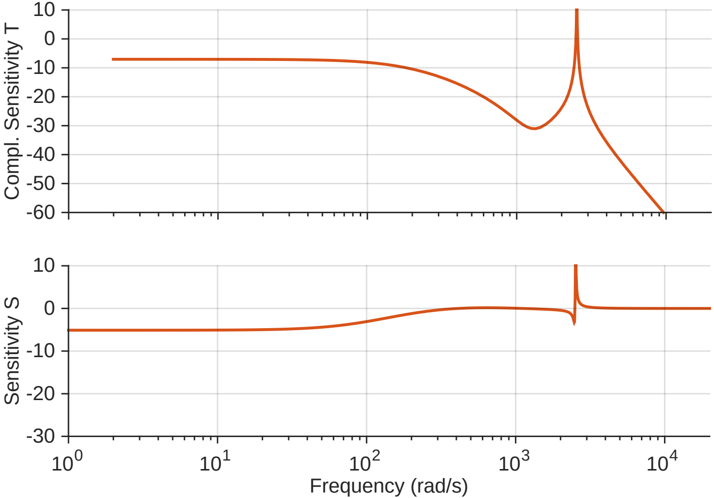
Goals

- Track ramp inputs
- Reduce sensitivity to noise at resonant frequency
- Minimize response time

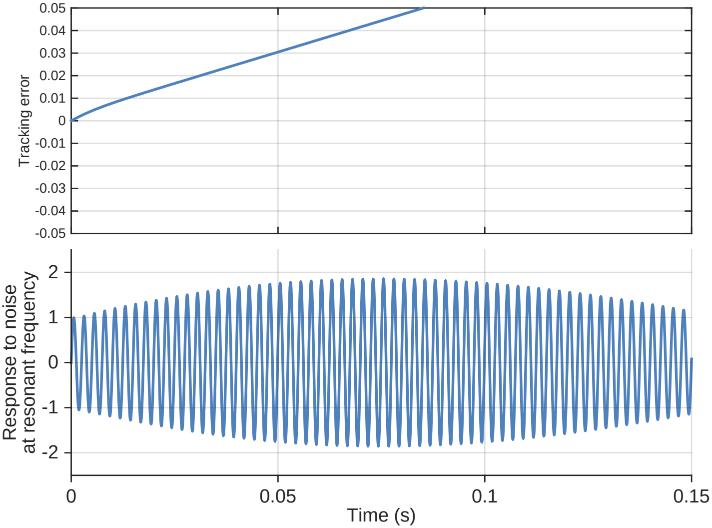
Frequency Response



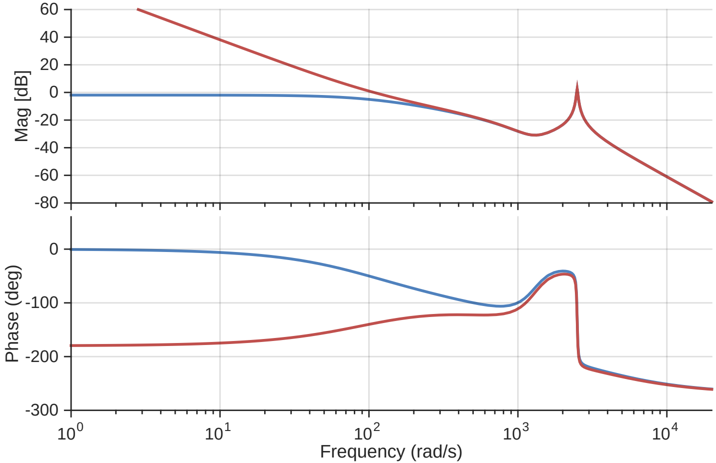
Closed-Loop Sensitivity



Step Response & Resonance Disturbance Rejection

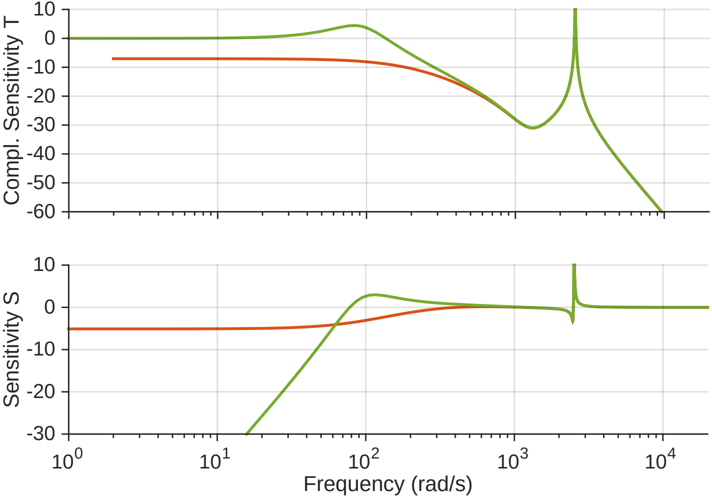


Frequency Response

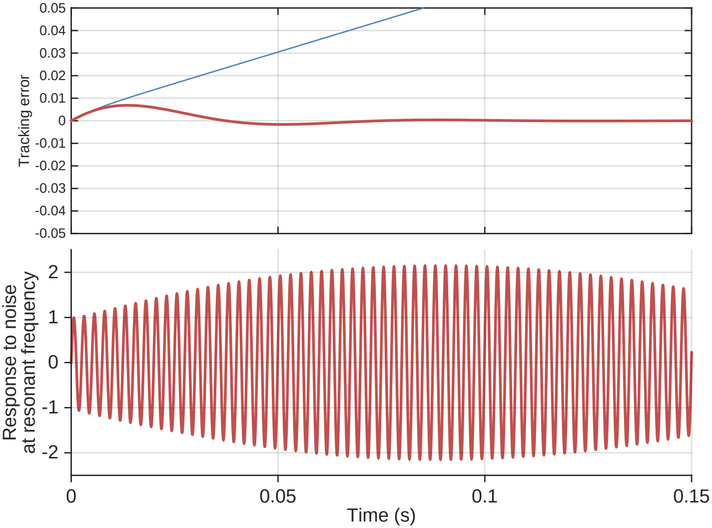


Track ramps $\rightarrow$ Add two integrators $\left(1 + \frac{1}{T_i s}\right)^2$

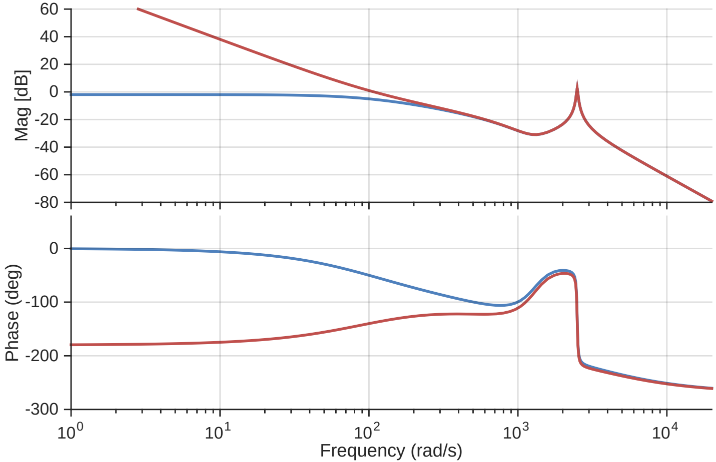
Closed-Loop Sensitivity



Step Response & Resonance Disturbance Rejection



Frequency Response

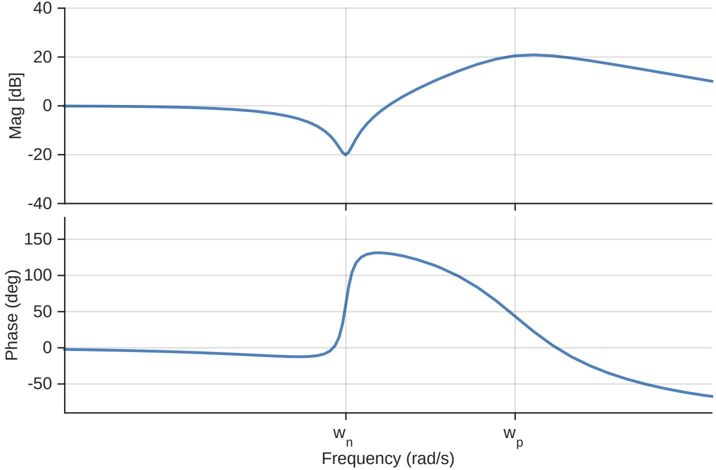


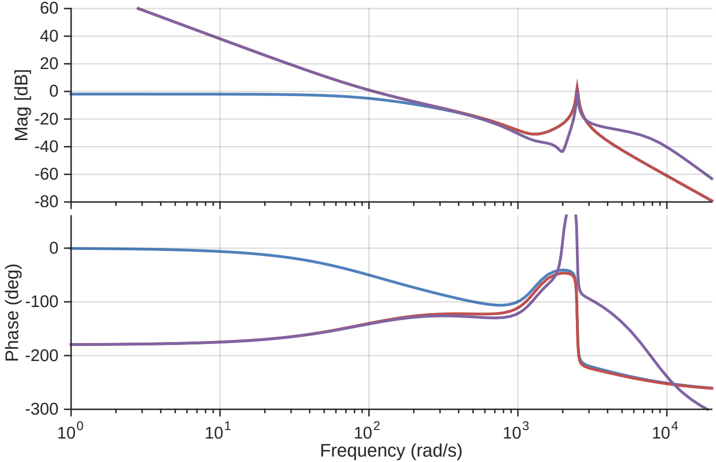
Reducing sensitivity at resonance, requires a ***high*** gain.

Problem: drop of 180 degrees of phase.

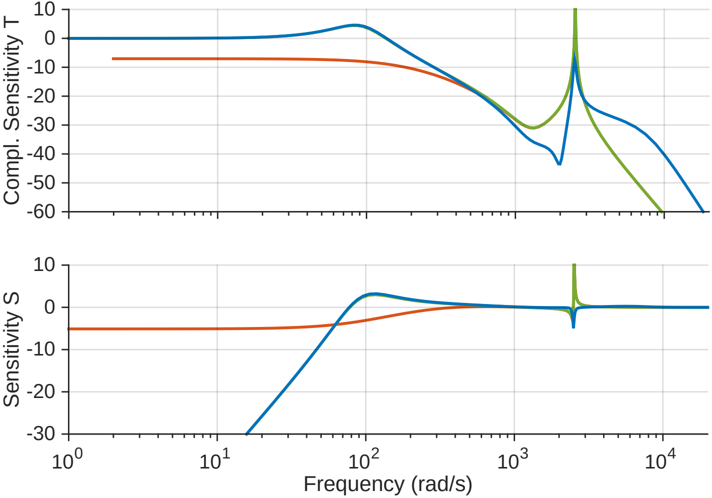
Notch Filter

$$N = \frac{(s/w_n)^2 + 2\zeta_n/w_n s + 1}{((s/w_p)^2 + 2\zeta_p/w_p s + 1)(s/p + 1)}$$

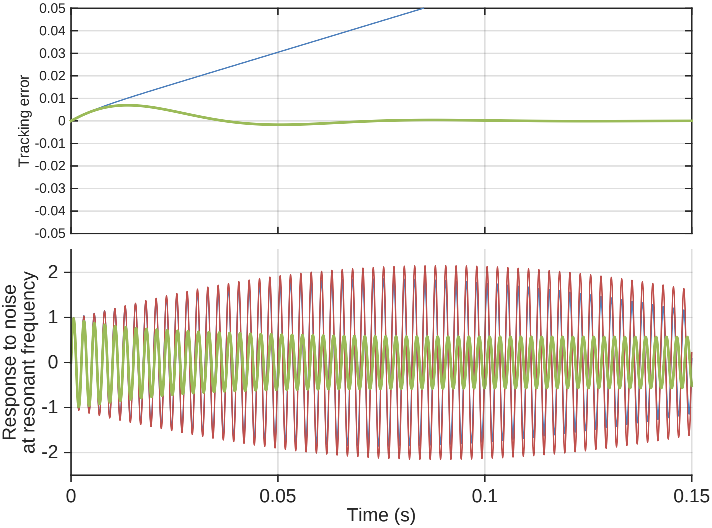



60
40
20
0
-20
-40
-60
-80
Mag [dB]
0
-100
-200
-300
Phase (deg)
10^0
10^1
10^2
10^3
10^4
Frequency (rad/s)

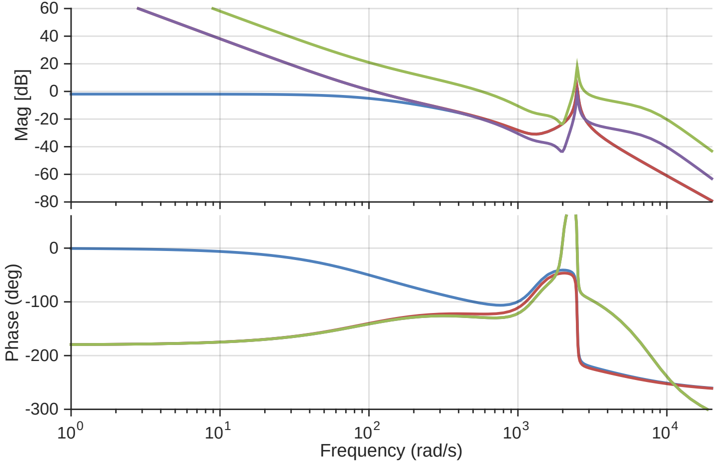
Closed-Loop Sensitivity



Step Response & Resonance Disturbance Rejection

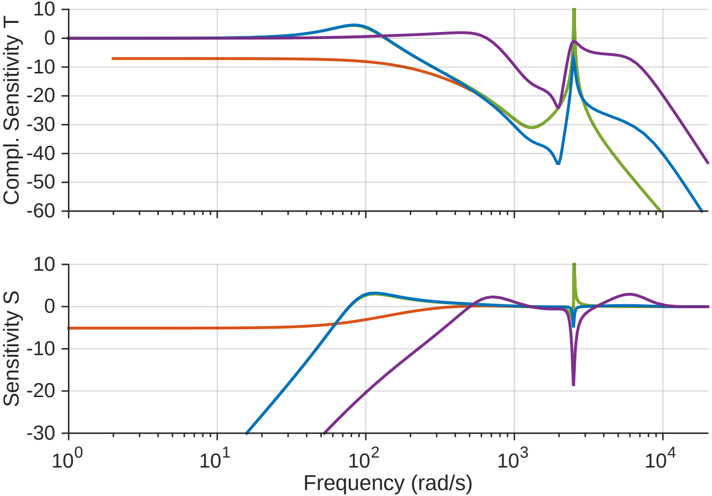


Frequency Response

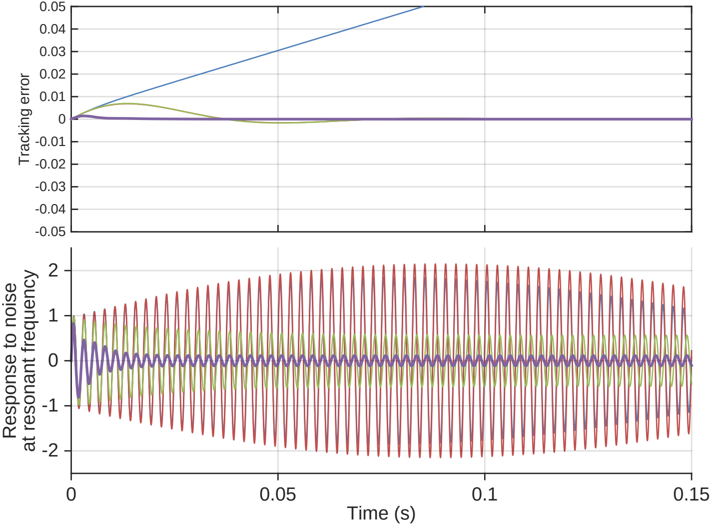


Increase gain to get best tracking performance.

Closed-Loop Sensitivity



Step Response & Resonance Disturbance Rejection



Result of a Scan